Polynomial Clones on Squarefree Groups

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Clones

Definition. A set of finitary functions C on a set A is a *clone* if

- C contains all projections $p_i^{(k)} : A^k \to A, (x_1, \ldots, x_k) \mapsto x_i$,
- C is closed under composition, $\forall f \in C_k, g_1, \ldots, g_k \in C_n$:

$$f(g_1, \ldots, g_k) : A^n \to A,\ \bar{x} \mapsto f(g_1(\bar{x}), \ldots, g_k(\bar{x})) \text{ is in } C_n.$$

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Some classical results.

clones on a finite set A:	$\aleph_0$ if $\lvert A \rvert = 2$, c else
clones containing all constants:	7 if $\lvert A \rvert = 2$, c else
constants and a Mal'cev operation:	finite iff $\lvert A \rvert \leq 3$ (Idziak, 1999)

Polynomial clones on groups

$\mathrm{Pol}(\mathbf{A})$ ... smallest clone that contains all constant functions and all operations F of an algebra $\mathbf{A} := \langle A, F \rangle$.

Examples.

- $\mathrm{Pol}(\mathbb{Z}_2, +)$ ... e.g. $(x_1, x_2) \mapsto x_1 + x_2 + 1$.
  $\mathrm{Pol}(\mathbb{Z}_2, +, \cdot)$ ... all (polynomial) functions.

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- Let $\mathbf{A}_k := \langle \mathbb{Z}_4, +, 2x_1 \dots x_k \rangle$.

$$\mathrm{Pol}(\mathbb{Z}_4, +) \subsetneq \mathrm{Pol}(\mathbf{A}_2) \subsetneq \mathrm{Pol}(\mathbf{A}_3) \subsetneq \dots$$

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Conjectures. (Idziak, 1999) Let n be squarefree.

1. Only finitely many clones contain $\mathrm{Pol}(\mathbb{Z}_n, +)$.
2. For $\mathbf{A} := \langle \mathbb{Z}_n, \{+\} \cup F \rangle$, $\mathrm{Pol}(\mathbf{A})$ is uniquely determined by $\mathrm{Con}(\mathbf{A}, \wedge, \vee, [.,.])$.

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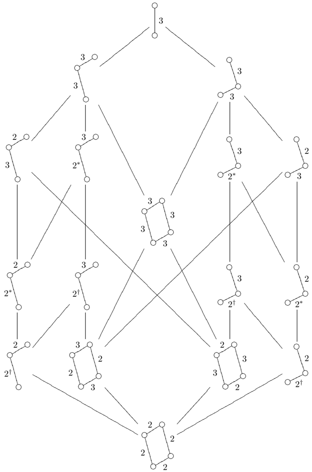
Results

Theorem. (PM, submitted 2007)
For an algebra $\mathbf{A}$ with squarefree group reduct, $\mathrm{Pol}(\mathbf{A})$ is determined by $\mathrm{Pol}_2(\mathbf{A})$.

Corollary. (PM, submitted 2007)
There are only finitely many clones on A that contain a group operation and all constants iff $|A|$ is squarefree.

Polynomial clones on $\langle \mathbb{Z}_{pq}, + \rangle$ (Aichinger, PM, 2007)



Outline of the proof of the Theorem.

Let **A** have a squarefree group reduct. Show

$$\mathrm{Pol}(\mathbf{A}) = \{f : A^k \to A \;\mid\; k \in \mathbb{N} \text{ and } f(g_1, \ldots, g_k) \in \mathrm{Pol}_2(\mathbf{A}) \; \forall g_1, \ldots, g_k \in \mathrm{Pol}_2(\mathbf{A})\}.$$

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1. Reduce to **A** with cyclic group reduct.
   [**Lemma.** For a group $\langle A, \cdot \rangle$ with cyclic Sylow subgroups there is $+$ in $\mathrm{Pol}_2(A, \cdot)$ such that $\langle A, + \rangle$ is a cyclic group.]

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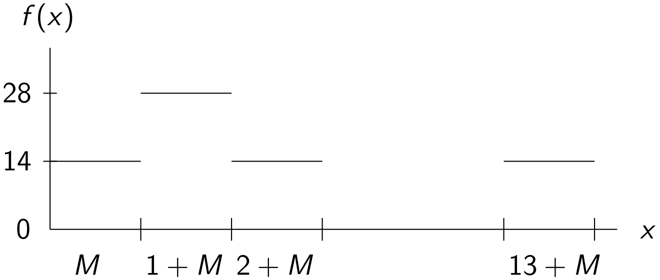
3. Determine the polynomial functions $A^k \to 0/\mu$.
4. Reconstruct $\mathrm{Pol}_k(\mathbf{A})$ from $\mathrm{Pol}_k(\mathbf{A}/\mu)$ and $\mathrm{Pol}_k(\mathbf{A}) \cap (0/\mu)^{A^k}$.

Piecewise constant functions into $0/\mu$

Assume $\mathbf{A} := \langle \mathbb{Z}_{42}, \{+\} \cup F \rangle$ has an abelian monolith μ with $M := 0/\mu$, $|M| = 3$.
$W := \{f : \mathbb{Z}_{42} \to M \mid f(x + M) = f(x)\ \forall x \in \mathbb{Z}_{42}\}$.



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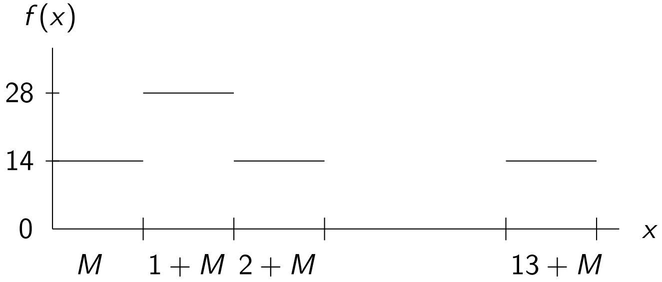

- Let $F := \mathrm{GF}(3)$. Then $W \cong F^{14}$.

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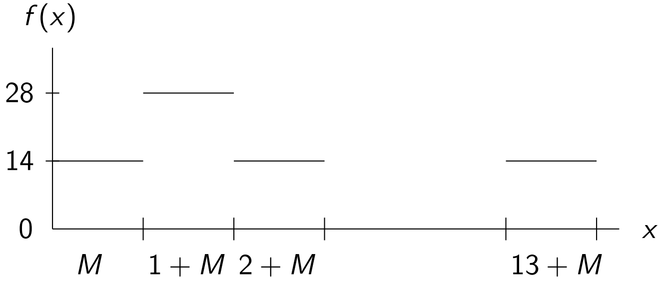

- Let $F := \mathrm{GF}(3)$. Then $W \cong F^{14}$.
- Let $G := \{\mathbb{Z}_{14} \to \mathbb{Z}_{14}, x \mapsto ax + b \mid \gcd(a, 14) = 1, b \in \mathbb{Z}_{14}\}$. W is an $F[G]$-module that is the sum of simple $F[G]$-modules.

Interlude: Representation theory

Lemma.
Let F, K be finite fields with distinct characteristics, let $k \in \mathbb{N}$. Then $F[K^k]$ splits into 2 simple $F[\mathrm{AGL}(k, K)]$-modules whose endomorphism algebras are F.

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- There exist $B_1, \ldots, B_l \leq \langle A, + \rangle$ such that

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- Similarly there exist $C_1, \ldots, C_n \leq \langle A, + \rangle$ such that

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- $\mathrm{Pol}_1(\mathbf{A}) \cap W$ is an $F[G]$-submodule of W.
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- $\{B_1, \dots, B_l\} = \{C_1, \dots, C_n\}$.

Congruences and commutators are not enough

Let $A := \mathbb{Z}_{210}$,

$$\mathbf{A}_1 := \langle A, +, g_6, g_{10}, g_{15} \rangle \text{ and } \mathbf{A}_2 := \langle A, +, g_{30} \rangle$$

with $g_r(rA) = 30$ and $g_r(A \setminus rA) = 0$.

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Then $\mathrm{Con}(\mathbf{A}_1, \wedge, \vee, [.,.]) = \mathrm{Con}(\mathbf{A}_2, \wedge, \vee, [.,.])$ but $\mathrm{Pol}(\mathbf{A}_1) \subsetneq \mathrm{Pol}(\mathbf{A}_2)$.

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$$\begin{aligned} R := \{(x_1, \ldots, x_8) \in A^8 \quad | \quad & \{x_2 - x_1, x_5 - x_3, x_7 - x_4, x_8 - x_6\} \subseteq 6A, \\ & \{x_3 - x_1, x_5 - x_2, x_6 - x_4, x_8 - x_7\} \subseteq 10A, \\ & \{x_4 - x_1, x_6 - x_3, x_7 - x_2, x_8 - x_5\} \subseteq 15A, \\ & x_1 - (x_2 + x_3 + x_4) + x_5 + x_6 + x_7 = x_8\} \end{aligned}$$

is preserved by $+, g_6, g_{10}, g_{15}$ but not by g_{30}.