On the number of polynomially inequivalent finite Mal'cev algebras with congruence lattice of height two.

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Basic Definitions

Polynomial functions (Pol $\mathbf{A}$)

Definition 1. Let $\mathbf{A} = \langle A; F\rangle$ be an algebra, and let $k \in \mathbb{N}$. Then Pol$_k\mathbf{A}$ is the subuniverse of $\mathbf{A}^{A^k}$ generated by

- the projection functions $\langle x_1, x_2, \ldots, x_k\rangle \mapsto x_i$ with $i = 1, \ldots, k$.
- the constant functions $\langle x_1, x_2, \ldots, x_k\rangle \mapsto a$ with $a \in A$.

Proposition 2. Let $\mathbf{A}$ be an algebra, and let $k \in \mathbb{N}$. Then $p : A^k \to A$ lies in $\mathsf{Pol}_k\mathbf{A}$ iff there is an $m \in \mathbb{N}$, a term function $t \in \mathsf{Clo}_{m+k}\mathbf{A}$, and there are elements $a_1, a_2, \ldots, a_m \in A$ such that

$$p(x_1, x_2, \ldots, x_k) = t(a_1, a_2, \ldots, a_m, x_1, x_2, \ldots, x_k)$$

for all $x_1, x_2, \ldots, x_k \in A$.

Clones

Definition 3 (Clones)**.** A subset of $\bigcup\{A^{A^i} \mid i \in \mathbb{N}\}$ is a *clone* on A if it contains all projections, and it is closed under all compositions of functions. It is *constantive* if it contains all constant unary operations.

Proposition 4. A subset C of $\bigcup\{A^{A^i} \mid i \in \mathbb{N}\}$ is a constantive clone on A iff there is a family of operations F on A such that

$$\mathrm{Pol}\,\langle A; F\rangle = C.$$

Constantive clones

Question: Given a finite set A, how many constantive clones do we have on A?

Answer: [Ágoston et al., 1986] seven if $|A| = 2$, $2^{\aleph_0}$ if $|A| \geq 3$.

Question: (R. McKenzie, oral communication) Given a finite set A, how many constantive clones do we have on A that contain a ternary Mal'cev operation?

Polynomial Equivalence

Definition 5. Let $\mathbf{A}_1 := \langle A; F_1 \rangle$ and $\mathbf{A}_2 := \langle A; F_2 \rangle$ be algebras. $\mathbf{A}_1$ and $\mathbf{A}_2$ are *polynomially equivalent* if and only if $\mathsf{Pol}\,\mathbf{A}_1 = \mathsf{Pol}\,\mathbf{A}_2$.

Theorem 6 (Ágoston, Demetrovics, Hannák, 1986)**.** Let A be a set with $|A| \geq 3$. Then there exist $2^{\aleph_0}$ polynomially inequivalent algebras with universe A.

Problem 7. Are there uncountably many finite polynomially inequivalent algebras with a Mal'cev polynomial?

. . . whose universe is an initial section of $\mathbb{N}$. . .

Constantive clones on expanded groups

Problem 8. Let $\mathbf{V} = \langle V; +, F\rangle$ be a finite expanded group. How many clones C do we have on V such that

$$\mathsf{Pol}\,\mathbf{V} \subseteq C?$$

Examples

Theorem 9 ([Bulatov and Idziak, 2003], [Idziak, 1999])**.** The algebras $\langle \mathbb{Z}_4; +\rangle$, $\langle \mathbb{Z}_4; +, 2xy\rangle$, $\langle \mathbb{Z}_4; +, 2xyz\rangle$, ... are pairwise polynomially inequivalent.

Conjecture 10 ([Idziak, 1999])**.** Let n be a squarefree natural number. Then $\langle \mathbb{Z}_n; +\rangle$ has only finitely many polynomially inequivalent expansions.

Theorem 11 ([Aichinger and Mayr, 2006])**.** Let p, q be primes with $p \neq q$. Then there are exactly 17 clones on $\mathbb{Z}_{pq}$ that contain Pol $\langle \mathbb{Z}_{pq}; + \rangle$.

Idea: on every expansion of $\langle \mathbb{Z}_{pq}; + \rangle$, every function that preserves commutators and congruences is polynomial.

Wild Conjecture Let $\mathbf{A}$ an algebra with a Mal'cev polynomial. Then there are at most countably many clones C with $\mathsf{Pol}\,\mathbf{A} \subseteq C$.

Wilder Conjecture Let $\mathbf{A}$ be an algebra with a Mal'cev polynomial m. Then there exists a finite set R of relations on A such that

$$\mathsf{Pol}\,\mathbf{A} = \mathsf{Comp}(A, R).$$

To disprove: find an infinite descending chain of clones that contain $\mathsf{Pol}\,(\langle A; m \rangle)$.

Theorem 12. Let $\mathbf{V} = \langle V; +, F \rangle$ be an expanded group. If $h(\mathsf{Con}\,\mathbf{V}) \leq 2$, then there is a finite set of relations R such that

$$\mathsf{Pol}\,\mathbf{V} = \mathsf{Comp}(V, R).$$

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