Some applications of higher commutators to Mal'cev algebras

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Outline

The term equivalence problem

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Let **A** be an algebra. We look for an algorithm solving the following problem:

- Given: Terms s, t in the language of **A**.
- Asked: Do s and t induce the same term function on **A**?

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Results about the term equivalence problem

Theorem (Burris, Lawrence, 2004)

Let **G** be a finite nilpotent group of class k, let $n \in \mathbb{N}$, and let $p \in \mathrm{Pol}_n\mathbf{G}$. If $p(a_1, \dots, a_n) = 1$ for all $(a_1, \dots, a_n)$ with

$$|\{i \in \{1, \dots, n\} \mid a_i \neq 1\}| \leq k,$$

then $p(\overline{x}) = 1$ for all $\overline{x} \in G^n$.

Theorem (Hunt, Stearns, 1990, Burris, Lawrence 1993)

For a finite nilpotent ring, term equivalence can be decided in polynomial time.

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Affine completeness

Definition

An algebra $\mathbf{A}$ is affine complete if $\operatorname{Pol}\mathbf{A} = \operatorname{Comp}(A, \operatorname{Con}\mathbf{A})$.

Theorem (EA, Ecker, 2006)

There is an algorithm that solves the following problem:

- Given: A finite nilpotent group $\mathbf{G}$.
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Clone theory

Theorem (Idziak, Bulatov)

$$\mathrm{Pol}\,\langle \mathbb{Z}_4, +, 2x_1x_2\rangle \subset \mathrm{Pol}\,\langle \mathbb{Z}_4, +, 2x_1x_2x_3\rangle \subset \mathrm{Pol}\,\langle \mathbb{Z}_4, +, 2x_1x_2x_3x_4\rangle \subset \cdots$$

Theorem (Bulatov, 2002)

For every clone C with $\mathrm{Pol}\,\langle \mathbb{Z}_4, +\rangle \subseteq C$, there is a finite set R of relations such that

$$C = \mathrm{Comp}\,(\mathbb{Z}_4, R).$$

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Higher commutators

Bulatov's multiplaced commutators (Proc. AAA60, 2001)

Let **A** be an algebra. For every $n \in \mathbb{N}$, A. Bulatov defined an n-ary operation on Con **A**

$$(\alpha_1, \ldots, \alpha_n) \mapsto [\alpha_1, \ldots, \alpha_n]$$

with the following properties:

- For $n = 2$, this operation yields the term-condition commutator.
- $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge\{\alpha_i \mid i \in \{1, \ldots, n\}\}$.
- $[\bullet, \bullet, \ldots, \bullet]$ is monotonous.
- $[\alpha_1, \ldots, \alpha_n] \leq [\alpha_1, \ldots, \alpha_{n-2}, \alpha_n]$.

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Properties of Bulatov's commutators

Theorem (Mudrinski, 2007)

Let **A** be an algebra with a Mal'cev term, and let $n \in \mathbb{N}$. Then:

- $[\bullet, \bullet, \dots, \bullet]$ is join distributive in each argument (w.r.t. arbitrary joins).
- $[\alpha_1, \dots, \alpha_n] = [\alpha_{\pi(1)}, \dots, \alpha_{\pi(n)}]$ for every $\pi \in S_n$. (This was already claimed by Bulatov.)
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Theorem (cf. Kearnes, 1999)

Let **A** be an algebra of finite type in a congruence modular variety. If **A** is nilpotent and of prime power cardinality, then there exists a $c \in \mathbb{N}$ such that

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Clone Theory

Theorem (EA, Mudrinski, 2007)

Let $\mathbf{A}$ be a finite Mal'cev algebra with congruence lattice of height 2. Then there is a finite set R of relations on A such that $\operatorname{Pol}\mathbf{A} = \operatorname{Comp}(A, R)$.

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