The surprising ubiquity of planar near-rings

Erhard Aichinger

Department of Algebra
Johannes Kepler University Linz, Austria

Combinatorics 2008, Costermano, Italy

Outline

Near-rings of functions

Functions on groups

Let $\langle G, +, -, 0\rangle$ be a group. We define

$$M(G) := G^G.$$

On $M(G)$, we define $\circ$ by

$$f \circ g\ (\gamma) = f(g(\gamma)) \text{ for all } f, g \in M(G), \gamma \in G.$$

Near-rings of functions

Properties of $M(G)$

- $\langle M(G), +, -, 0 \rangle$ is a group (the direct product $\mathbf{G}^G$).
- $\langle M(G), \circ \rangle$ is a semigroup.
- For all $f, g, h \in M(G)$:

$$(f + g) \circ h = f \circ h + g \circ h.$$

- If $|G| > 1$, then there are $f, g, h \in M(G)$ such that

$$f \circ (g + h) \neq f \circ g + f \circ h.$$

- For all $f \in M(G)$: $0 \circ f = 0$.

Near-rings of functions

Properties of $M(G)$

- $\langle M(G), +, -, 0\rangle$ is a group (the direct product $\mathbf{G}^G$).
- $\langle M(G), \circ\rangle$ is a semigroup.
- For all $f, g, h \in M(G)$:

$$(f+g)\circ h = f\circ h + g\circ h.$$

- If $|G| > 1$, then there are $f, g, h \in M(G)$ such that

$$f\circ(g+h) \neq f\circ g + f\circ h.$$

- For all $f \in M(G)$: $0 \circ f = 0$.

Near-rings of functions

Properties of $M(G)$

- $\langle M(G), +, -, 0 \rangle$ is a group (the direct product $\mathbf{G}^G$).
- $\langle M(G), \circ \rangle$ is a semigroup.
- For all $f, g, h \in M(G)$:

$$(f+g) \circ h = f \circ h + g \circ h.$$

- If $|G| > 1$, then there are $f, g, h \in M(G)$ such that

$$f \circ (g+h) \neq f \circ g + f \circ h.$$

- For all $f \in M(G)$: $0 \circ f = 0$.

Near-rings of functions

Properties of $M(G)$

- $\langle M(G), +, -, 0\rangle$ is a group (the direct product $\mathbf{G}^G$).
- $\langle M(G), \circ\rangle$ is a semigroup.
- For all $f, g, h \in M(G)$:

$$(f+g)\circ h = f\circ h + g\circ h.$$

- If $|G| > 1$, then there are $f, g, h \in M(G)$ such that

$$f\circ(g+h) \neq f\circ g + f\circ h.$$

- For all $f \in M(G)$: $0 \circ f = 0$.

Near-rings of functions

Properties of $M(G)$

- $\langle M(G), +, -, 0\rangle$ is a group (the direct product $\mathbf{G}^G$).
- $\langle M(G), \circ\rangle$ is a semigroup.
- For all $f, g, h \in M(G)$:

$$(f+g)\circ h = f\circ h + g\circ h.$$

- If $|G| > 1$, then there are $f, g, h \in M(G)$ such that

$$f\circ(g+h) \neq f\circ g + f\circ h.$$

- For all $f \in M(G)$: $0\circ f = 0$.

Near-rings of functions

Properties of $M(G)$

- $\langle M(G), +, -, 0\rangle$ is a group (the direct product $\mathbf{G}^G$).
- $\langle M(G), \circ\rangle$ is a semigroup.
- For all $f, g, h \in M(G)$:

$$(f+g) \circ h = f \circ h + g \circ h.$$

- If $|G| > 1$, then there are $f, g, h \in M(G)$ such that

$$f \circ (g+h) \neq f \circ g + f \circ h.$$

- For all $f \in M(G)$: $0 \circ f = 0$.

Definition of Near-rings

A near-ring is an algebra **N** with operations $+$ (binary), $-$ (unary), 0 (nullary), and $\circ$ (binary) such that

- $\langle N, +, -, 0 \rangle$ is a (not necessarily abelian) group,
- **N** satisfies the identity $(x + y) \circ z \approx x \circ z + y \circ z$, and
- **N** satisfies the identity $(x \circ y) \circ z \approx x \circ (y \circ z)$.

Definition of Near-rings

A near-ring is an algebra **N** with operations $+$ (binary), $-$ (unary), 0 (nullary), and $\circ$ (binary) such that

- $\langle N, +, -, 0\rangle$ is a (not necessarily abelian) group,
- **N** satisfies the identity $(x + y) \circ z \approx x \circ z + y \circ z$, and
- **N** satisfies the identity $(x \circ y) \circ z \approx x \circ (y \circ z)$.

Definition of Near-rings

A near-ring is an algebra **N** with operations $+$ (binary), $-$ (unary), 0 (nullary), and $\circ$ (binary) such that

- $\langle N, +, -, 0\rangle$ is a (not necessarily abelian) group,
- **N** satisfies the identity $(x + y) \circ z \approx x \circ z + y \circ z$, and
- **N** satisfies the identity $(x \circ y) \circ z \approx x \circ (y \circ z)$.

Definition of Near-rings

A near-ring is an algebra **N** with operations $+$ (binary), $-$ (unary), 0 (nullary), and $\circ$ (binary) such that

- $\langle N, +, -, 0\rangle$ is a (not necessarily abelian) group,
- **N** satisfies the identity $(x + y) \circ z \approx x \circ z + y \circ z$, and
- **N** satisfies the identity $(x \circ y) \circ z \approx x \circ (y \circ z)$.

Near-fields

Definition of Near-fields

A near-field is a near-ring $\langle N, +, \circ \rangle$ such that $\langle N \setminus \{0\}, \circ \rangle$ is a group.

The Quaternion Near-field with 9 Elements

We construct a 9-element near-field using the terminology of the "Ferrero-Near-ring-Factory" (Clay). (cf. Ferrero, Classificazione e costruzione degli stems p-singolari, Ist. Lombardo, 1968)

Let $G := \mathbb{Z}_3 \times \mathbb{Z}_3$, and let $\Phi := \langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right) \rangle \leq \mathrm{GL}(2, \mathbb{Z}_3)$. Define

$$a \circ b := M(b) \cdot a,$$

where $M(b)$ is the matrix in Φ with first column b.

Then $\langle \mathbb{Z}_3 \times \mathbb{Z}_3, +, \circ \rangle$ is a near-field with multiplicative group Q_8.

Near-fields

Definition of Near-fields

A near-field is a near-ring $\langle N, +, \circ \rangle$ such that $\langle N \setminus \{0\}, \circ \rangle$ is a group.

The Quaternion Near-field with 9 Elements

We construct a 9-element near-field using the terminology of the “Ferrero-Near-ring-Factory” (Clay). (cf. Ferrero, Classificazione e costruzione degli stems p-singolari, Ist. Lombardo, 1968)

Let $G := \mathbb{Z}_3 \times \mathbb{Z}_3$, and let $\Phi := \langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right) \rangle \leq \mathrm{GL}(2, \mathbb{Z}_3)$. Define

$$a \circ b := M(b) \cdot a,$$

where $M(b)$ is the matrix in Φ with first column b. Then $\langle \mathbb{Z}_3 \times \mathbb{Z}_3, +, \circ \rangle$ is a near-field with multiplicative group Q_8.

Near-fields in Geometry

Near-field elements as coordinates

Finite projective planes of Lenz-Barlotti-types IV.a.2 and IV.a.3 can be coordinatized by near-fields.

Rings satisfying $x^n = x$

Theorem (Jacobson, 1945)

Let $n > 1$. Every ring satisfying $x^n = x$ is a subdirect product of fields satisfying $x^n = x$.

A consequence in equational logic

$$
\begin{aligned}
&(x + y) + z = x + (y + z)\\
&0 + x = x\\
&-x + x = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x + y = y + x\\
&(xy)z = x(yz) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow\ xy = yx\\
&(x + y)z = xz + yz\\
&x(y + z) = xy + xz\\
&x^n = x
\end{aligned}
$$

Rings satisfying $x^n = x$

Theorem (Jacobson, 1945)

Let $n > 1$. Every ring satisfying $x^n = x$ is a subdirect product of fields satisfying $x^n = x$.

A consequence in equational logic

$$
\begin{array}{ll}
(x+y)+z = x+(y+z) & \\
0+x = x & \\
-x+x = 0 & \\
 & \quad x+y = y+x \\
(xy)z = x(yz) & \Rightarrow \; xy = yx \\
(x+y)z = xz+yz & \\
x(y+z) = xy+xz & \\
x^n = x &
\end{array}
$$

Near-rings satisfying $x^n = x$

Theorem (S. Ligh, Kyungpook Math J., 1971)

Let $n > 1$. Every **zerosymmetric near-**ring ($x \circ 0 \approx 0$) **with left identity** ($x \approx 1 \circ x$) satisfying $x^n = x$ is a subdirect product of **near-**fields satisfying $x^n = x$.

$(x + y) + z = x + (y + z)$
$0 + x = x$
$-x + x = 0$
$(xy)z = x(yz)$ $\quad \Rightarrow \quad x + y = y + x$
$(x + y)z = xz + yz$ $\quad ?\ xy = yx\ ?$
$x0 = 0$
$1x = x$
$x^n = x$

Near-rings satisfying $x^n = x$

Theorem (S. Ligh, Kyungpook Math J., 1971)

Let $n > 1$. Every **zerosymmetric near-**ring ($x \circ 0 \approx 0$) **with left identity** ($x \approx 1 \circ x$) satisfying $x^n = x$ is a subdirect product of **near-**fields satisfying $x^n = x$.

$(x + y) + z = x + (y + z)$
$0 + x = x$
$-x + x = 0$
$(xy)z = x(yz)$ $\qquad \Rightarrow \; x + y = y + x$
$(x + y)z = xz + yz$ $\qquad ?\; xy = yx \;?$
$x0 = 0$
$1x = x$
$x^n = x$

Near-rings satisfying $x^n = x$

Theorem (S. Ligh, Kyungpook Math J., 1971)

Let $n > 1$. Every **zerosymmetric near-**ring ($x \circ 0 \approx 0$) **with left identity** ($x \approx 1 \circ x$) satisfying $x^n = x$ is a subdirect product of **near-**fields satisfying $x^n = x$.

$$
\begin{array}{ll}
(x+y)+z = x+(y+z) & \\
0+x = x & \\
-x+x = 0 & \\
(xy)z = x(yz) & \Rightarrow\ x+y = y+x \\
(x+y)z = xz+yz & \quad ?\ xy = yx\ ? \\
x0 = 0 & \\
1x = x & \\
x^n = x &
\end{array}
$$

Zero-symmetric Near-rings with Left Identity

Let $\mathcal{V}$ denote the variety of zerosymmetric near-rings with left identity.

Corollaries of Ligh's result

Let $n \in \mathbb{N} \setminus \{1\}$, let $R \in \mathcal{V}$ satisfy $x^n \approx x$.

1. Then R satisfies $x + y = y + x$.
2. If every near-field whose multiplicative exponent divides $n - 1$ is a field, then R satisfies $xy = yx$.

Zero-symmetric Near-rings with Left Identity

Let $\mathcal{V}$ denote the variety of zerosymmetric near-rings with left identity.

Corollaries of Ligh's result

Let $n \in \mathbb{N} \setminus \{1\}$, let $R \in \mathcal{V}$ satisfy $x^n \approx x$.

1. Then R satisfies $x + y = y + x$.
2. If every near-field whose multiplicative exponent divides $n - 1$ is a field, then R satisfies $xy = yx$.

Zero-symmetric Near-rings with Left Identity

Let $\mathcal{V}$ denote the variety of zerosymmetric near-rings with left identity.

Corollaries of Ligh's result

Let $n \in \mathbb{N} \setminus \{1\}$, let $R \in \mathcal{V}$ satisfy $x^n \approx x$.

1. Then R satisfies $x + y = y + x$.
2. If every near-field whose multiplicative exponent divides $n - 1$ is a field, then R satisfies $xy = yx$.

Zero-symmetric Near-rings with Left Identity

Let $\mathcal{V}$ denote the variety of zerosymmetric near-rings with left identity.

Corollaries of Ligh's result

Let $n \in \mathbb{N} \setminus \{1\}$, let $R \in \mathcal{V}$ satisfy $x^n \approx x$.

1. Then R satisfies $x + y = y + x$.
2. If every near-field whose multiplicative exponent divides $n - 1$ is a field, then R satisfies $xy = yx$.

Near-fields with finite multiplicative exponent

Theorem (Suchkov, Algebra and Logic, 2001)

Let F be near-field whose multiplicative group is a 2-group. Then $F = D_9$ or F is a finite field.

Theorem (Jabara, J. Austr. Math. Soc., 2004)

$\mathbf{GF}(2)$ is the only near-field satisfying $x^6 = x$.

Theorem (Jabara, Mayr, Forum Mathematicum, 2008, in print)

Let F be a near-field with multiplicative exponent $2^m 3^n$ for some $m \geq 0$ and $n \in \{0, 1, 2\}$. Then $|F| \in \{3^2, 5^2, 7^2, 17^2\}$ or F is a finite field.

Near-fields with finite multiplicative exponent

Theorem (Suchkov, Algebra and Logic, 2001)

Let F be near-field whose multiplicative group is a 2-group. Then $F = D_9$ or F is a finite field.

Theorem (Jabara, J. Austr. Math. Soc., 2004)

GF(2) is the only near-field satisfying $x^6 = x$.

Theorem (Jabara, Mayr, Forum Mathematicum, 2008, in print)

Let F be a near-field with multiplicative exponent $2^m 3^n$ for some $m \geq 0$ and $n \in \{0, 1, 2\}$. Then $|F| \in \{3^2, 5^2, 7^2, 17^2\}$ or F is a finite field.

Near-fields with finite multiplicative exponent

Theorem (Suchkov, Algebra and Logic, 2001)

Let F be near-field whose multiplicative group is a 2-group. Then $F = D_9$ or F is a finite field.

Theorem (Jabara, J. Austr. Math. Soc., 2004)

$\mathbf{GF}(2)$ is the only near-field satisfying $x^6 = x$.

Theorem (Jabara, Mayr, Forum Mathematicum, 2008, in print)

Let F be a near-field with multiplicative exponent $2^m 3^n$ for some $m \geq 0$ and $n \in \{0, 1, 2\}$. Then $|F| \in \{3^2, 5^2, 7^2, 17^2\}$ or F is a finite field.

The Equality $x^n \approx x$ in Near-rings with Left Identity

Zero-symmetric Near-rings with Left Identity that satisfy $x^n \approx x$

For $n \in \mathbb{N}$, let $\mathcal{V}_n$ denote the subvariety of $\mathcal{V}$ defined by $x^n \approx x$.

Corollary of the Theorems by Jabara and Mayr

For $n \in \{2, 3, 4, 6, 7, 10, 19\}$, all elements of $\mathcal{V}_n$ are commutative rings.

Question

Is there an infinite near-field whose multiplicative group has finite exponent?

The Equality $x^n \approx x$ in Near-rings with Left Identity

Zero-symmetric Near-rings with Left Identity that satisfy $x^n \approx x$

For $n \in \mathbb{N}$, let $\mathcal{V}_n$ denote the subvariety of $\mathcal{V}$ defined by $x^n \approx x$.

Corollary of the Theorems by Jabara and Mayr

For $n \in \{2, 3, 4, 6, 7, 10, 19\}$, all elements of $\mathcal{V}_n$ are commutative rings.

Question

Is there an infinite near-field whose multiplicative group has finite exponent?

The Equality $x^n \approx x$ in Near-rings with Left Identity

Zero-symmetric Near-rings with Left Identity that satisfy $x^n \approx x$

For $n \in \mathbb{N}$, let $\mathcal{V}_n$ denote the subvariety of $\mathcal{V}$ defined by $x^n \approx x$.

Corollary of the Theorems by Jabara and Mayr

For $n \in \{2, 3, 4, 6, 7, 10, 19\}$, all elements of $\mathcal{V}_n$ are commutative rings.

Question

Is there an infinite near-field whose multiplicative group has finite exponent?

Definition of Planar Near-rings

On a (right) near-ring **N**, we define an equivalence relation $\equiv$ by

$$a \equiv b :\Leftrightarrow \forall x \in N : x \circ a = x \circ b.$$

Definition (Anshel, Clay, Ferrero)

Let **N** be a near-ring. **N** is a planar near-ring: $\Leftrightarrow$

P1 N has at least 3 equivalence classes modulo $\equiv$, i.e. $|N/_{\equiv}| \geq 3$.

P2 For all $a, b, c \in N$ such that $a \not\equiv b$, there exists *precisely one* x such that

$$x \circ a = x \circ b + c.$$

The conditions P1 and P2 are called the *planarity conditions*.

Definition of Planar Near-rings

On a (right) near-ring **N**, we define an equivalence relation $\equiv$ by

$$a \equiv b :\Leftrightarrow \forall x \in N : x \circ a = x \circ b.$$

Definition (Anshel, Clay, Ferrero)

Let **N** be a near-ring. **N** is a planar near-ring: $\Leftrightarrow$

P1 N has at least 3 equivalence classes modulo $\equiv$, i.e. $|N/_{\equiv}| \geq 3$.

P2 For all $a, b, c \in N$ such that $a \not\equiv b$, there exists *precisely one* x such that

$$x \circ a = x \circ b + c.$$

The conditions P1 and P2 are called the *planarity conditions*.

Definition of Planar Near-rings

On a (right) near-ring **N**, we define an equivalence relation $\equiv$ by

$$a \equiv b :\Leftrightarrow \forall x \in N : x \circ a = x \circ b.$$

Definition (Anshel, Clay, Ferrero)

Let **N** be a near-ring. **N** is a planar near-ring: $\Leftrightarrow$

- P1 N has at least 3 equivalence classes modulo $\equiv$, i.e. $|N/_\equiv| \geq 3$.
- P2 For all $a, b, c \in N$ such that $a \not\equiv b$, there exists *precisely one* x such that

$$x \circ a = x \circ b + c.$$

The conditions P1 and P2 are called the *planarity conditions*.

Definition

Let **N** be a near-ring. **N** is a Ferrero near-ring if it satisfies P2. This means that for all $a, b, c \in N$ such that $a \not\equiv b$, there exists *precisely one* x such that

$$x \circ a = x \circ b + c.$$

Integral planar near-rings

Definition

A near-ring **N** is called integral iff for all $x, y \in N \setminus \{0\}$, we have $x \circ y \neq 0$.

Theorem (S. Ligh, 1971)

Let **N** be a finite, zero-symmetric near-ring. Then the following conditions are equivalent:

1. **N** is subdirectly irreducible, and for every $x \in N$, there is a natural number $n(x)$ such that $x^{n(x)} = x$.
2. **N** is subdirectly irreducible, and there is a natural number n such that for all $x \in N$, we have $x^n = x$.
3. **N** is an integral planar near-ring, or we have $a \circ b = a$ for all $a, b \in N$ with $b \neq 0$.
4. **N** is an integral Ferrero near-ring.

Integral planar near-rings

Definition

A near-ring **N** is called integral iff for all $x, y \in N \setminus \{0\}$, we have $x \circ y \neq 0$.

Theorem (S. Ligh, 1971)

Let **N** be a finite, zero-symmetric near-ring. Then the following conditions are equivalent:

1. **N** is subdirectly irreducible, and for every $x \in N$, there is a natural number $n(x)$ such that $x^{n(x)} = x$.
2. **N** is subdirectly irreducible, and there is a natural number n such that for all $x \in N$, we have $x^n = x$.
3. **N** is an integral planar near-ring, or we have $a \circ b = a$ for all $a, b \in N$ with $b \neq 0$.
4. **N** is an integral Ferrero near-ring.

Integral planar near-rings

Definition

A near-ring **N** is called integral iff for all $x, y \in N \setminus \{0\}$, we have $x \circ y \neq 0$.

Theorem (S. Ligh, 1971)

Let **N** be a finite, zero-symmetric near-ring. Then the following conditions are equivalent:

1. **N** is subdirectly irreducible, and for every $x \in N$, there is a natural number $n(x)$ such that $x^{n(x)} = x$.
2. **N** is subdirectly irreducible, and there is a natural number n such that for all $x \in N$, we have $x^n = x$.
3. **N** is an integral planar near-ring, or we have $a \circ b = a$ for all $a, b \in N$ with $b \neq 0$.
4. **N** is an integral Ferrero near-ring.

Integral planar near-rings

Definition

A near-ring **N** is called integral iff for all $x, y \in N \setminus \{0\}$, we have $x \circ y \neq 0$.

Theorem (S. Ligh, 1971)

Let **N** be a finite, zero-symmetric near-ring. Then the following conditions are equivalent:

1. **N** is subdirectly irreducible, and for every $x \in N$, there is a natural number $n(x)$ such that $x^{n(x)} = x$.
2. **N** is subdirectly irreducible, and there is a natural number n such that for all $x \in N$, we have $x^n = x$.
3. **N** is an integral planar near-ring, or we have $a \circ b = a$ for all $a, b \in N$ with $b \neq 0$.
4. **N** is an integral Ferrero near-ring.

Integral planar near-rings

Definition

A near-ring **N** is called integral iff for all $x, y \in N \setminus \{0\}$, we have $x \circ y \neq 0$.

Theorem (S. Ligh, 1971)

Let **N** be a finite, zero-symmetric near-ring. Then the following conditions are equivalent:

1. **N** is subdirectly irreducible, and for every $x \in N$, there is a natural number $n(x)$ such that $x^{n(x)} = x$.
2. **N** is subdirectly irreducible, and there is a natural number n such that for all $x \in N$, we have $x^n = x$.
3. **N** is an integral planar near-ring, or we have $a \circ b = a$ for all $a, b \in N$ with $b \neq 0$.
4. **N** is an integral Ferrero near-ring.

Integral planar near-rings

Definition

A near-ring **N** is called integral iff for all $x, y \in N \setminus \{0\}$, we have $x \circ y \neq 0$.

Theorem (S. Ligh, 1971)

Let **N** be a finite, zero-symmetric near-ring. Then the following conditions are equivalent:

1. **N** is subdirectly irreducible, and for every $x \in N$, there is a natural number $n(x)$ such that $x^{n(x)} = x$.
2. **N** is subdirectly irreducible, and there is a natural number n such that for all $x \in N$, we have $x^n = x$.
3. **N** is an integral planar near-ring, or we have $a \circ b = a$ for all $a, b \in N$ with $b \neq 0$.
4. **N** is an integral Ferrero near-ring.

Corollary

A finite zero-symmetric near-ring that satisfies $x^n \approx x$ is a direct product of integral Ferrero near-rings.

The variety of all near-rings

Known facts

Let $\mathcal{N}$ be the variety of all near-rings.

- Is $\mathcal{N}$ locally finite? No $(\langle \mathbb{Z}, +, -, 0, \cdot \rangle)$.
- Is $\mathcal{N}$ residually small? No ($\mathbf{M}(\mathbf{G})$ is simple if $|G| \neq 2$).
- Is the word problem for $\mathbf{F}_{\mathcal{N}}(X)$ solvable? Yes. We give a confluent, terminating rewrite system.

The variety of all near-rings

Known facts

Let $\mathcal{N}$ be the variety of all near-rings.

- Is $\mathcal{N}$ locally finite? No ($\langle \mathbb{Z}, +, -, 0, \cdot \rangle$).
- Is $\mathcal{N}$ residually small? No ($\mathbf{M}(\mathbf{G})$ is simple if $|G| \neq 2$).
- Is the word problem for $\mathbf{F}_{\mathcal{N}}(X)$ solvable? Yes. We give a confluent, terminating rewrite system.

The variety of all near-rings

Known facts

Let $\mathcal{N}$ be the variety of all near-rings.

- Is $\mathcal{N}$ locally finite? No ($\langle \mathbb{Z}, +, -, 0, \cdot \rangle$).
- Is $\mathcal{N}$ residually small? No ($\mathbf{M}(\mathbf{G})$ is simple if $|G| \neq 2$).
- Is the word problem for $\mathbf{F}_{\mathcal{N}}(X)$ solvable? Yes. We give a confluent, terminating rewrite system.

The variety of all near-rings

Known facts

Let $\mathcal{N}$ be the variety of all near-rings.

- Is $\mathcal{N}$ locally finite? No ($\langle \mathbb{Z}, +, -, 0, \cdot \rangle$).
- Is $\mathcal{N}$ residually small? No ($\mathbf{M}(\mathbf{G})$ is simple if $|G| \neq 2$).
- Is the word problem for $\mathbf{F}_{\mathcal{N}}(X)$ solvable? Yes. We give a confluent, terminating rewrite system.

A confluent rewrite system for near-rings

$$
\begin{aligned}
1 &: \quad 0 + x \to x \\
2 &: \quad -(x) + x \to 0 \\
3 &: \quad (x + y) + z \to x + (y + z) \\
4 &: \quad (x * y) * z \to x * (y * z) \\
5 &: \quad (x + y) * z \to (x * z) + (y * z) \\
6 &: \quad -(x) + (x + z) \to z \\
9 &: \quad -(0) \to 0 \\
110 &: \quad x + 0 \to x \\
12 &: \quad -(-(x)) \to x \\
13 &: \quad x + -(x) \to 0 \\
14 &: \quad x + (-(x) + z) \to z \\
16 &: \quad 0 * z \to 0 \\
19 &: \quad -(x + y) \to -(y) + -(x) \\
21 &: \quad -(x) * z \to -(x * z)
\end{aligned}
$$

More questions about varieties of near-rings

- Is the word problem for $\mathbf{F}_{\mathcal{A}}(X)$ is solvable, where $\mathcal{A}$ is the variety of all near-rings satisfying the identity $x + y \approx y + x$? YES.
- Is $\mathcal{N}$ generated by its finite members? YES.
- Is $\mathcal{A}$ generated by its finite members? YES.

More questions about varieties of near-rings

- Is the word problem for $\mathbf{F}_{\mathcal{A}}(X)$ is solvable, where $\mathcal{A}$ is the variety of all near-rings satisfying the identity $x + y \approx y + x$? YES.
- Is $\mathcal{N}$ generated by its finite members? YES.
- Is $\mathcal{A}$ generated by its finite members? YES.

More questions about varieties of near-rings

- Is the word problem for $\mathbf{F}_{\mathcal{A}}(X)$ is solvable, where $\mathcal{A}$ is the variety of all near-rings satisfying the identity $x + y \approx y + x$? YES.
- Is $\mathcal{N}$ generated by its finite members? YES.
- Is $\mathcal{A}$ generated by its finite members? YES.

Varieties that are generated by their finite members

The following varieties are generated by their finite members:

- The variety of all groups.
- The variety of all sets.
- The variety of all semigroups.

Varieties that are generated by their finite members

The following varieties are generated by their finite members:

- The variety of all groups.
- The variety of all sets.
- The variety of all semigroups.

Varieties that are generated by their finite members

The following varieties are generated by their finite members:

- The variety of all groups.
- The variety of all sets.
- The variety of all semigroups.

Varieties that are generated by their finite members

The following varieties are generated by their finite members:

- The variety of all groups.
- The variety of all sets.
- The variety of all semigroups.

Varieties that are not generated by their finite members

The following varieties are not generated by their finite members:

- The variety of vector-spaces over the rationals.
- Each variety that has an infinite member, but all of its finite members have precisely one element.
- The variety of modular lattices (Freese, Transactions AMS, 1979).
- The variety $\mathcal{B}_{665}$ of all groups of exponent dividing 665 (Shumyatsky, Journal of Pure and Applied Algebra, 2002).

Varieties that are not generated by their finite members

The following varieties are not generated by their finite members:

- The variety of vector-spaces over the rationals.
- Each variety that has an infinite member, but all of its finite members have precisely one element.
- The variety of modular lattices (Freese, Transactions AMS, 1979).
- The variety $\mathcal{B}_{665}$ of all groups of exponent dividing 665 (Shumyatsky, Journal of Pure and Applied Algebra, 2002).

Varieties that are not generated by their finite members

The following varieties are not generated by their finite members:

- The variety of vector-spaces over the rationals.
- Each variety that has an infinite member, but all of its finite members have precisely one element.
- The variety of modular lattices (Freese, Transactions AMS, 1979).
- The variety $\mathcal{B}_{665}$ of all groups of exponent dividing 665 (Shumyatsky, Journal of Pure and Applied Algebra, 2002).

Varieties that are not generated by their finite members

The following varieties are not generated by their finite members:

- The variety of vector-spaces over the rationals.
- Each variety that has an infinite member, but all of its finite members have precisely one element.
- The variety of modular lattices (Freese, Transactions AMS, 1979).
- The variety $\mathcal{B}_{665}$ of all groups of exponent dividing 665 (Shumyatsky, Journal of Pure and Applied Algebra, 2002).

Varieties that are not generated by their finite members

The following varieties are not generated by their finite members:

- The variety of vector-spaces over the rationals.
- Each variety that has an infinite member, but all of its finite members have precisely one element.
- The variety of modular lattices (Freese, Transactions AMS, 1979).
- The variety $\mathcal{B}_{665}$ of all groups of exponent dividing 665 (Shumyatsky, Journal of Pure and Applied Algebra, 2002).

Varieties that are generated by their finite members

Observation

If a variety $\mathcal{V}$ is generated by its finite members, then $\mathbf{F}_{\mathcal{V}}(X)$ is a subdirect product of finite algebras.

Observation

If $\mathcal{V}$ is generated by its finite members, $\mathcal{V}$ is a variety of algebras with finitely many operation symbols, and $\mathcal{V}$ is defined by finitely many identities, then for every set X, the word problem for the free algebra $\mathbf{F}_{\mathcal{V}}(X)$ is solvable (Evans, Bull. Amer. Math. Soc., 1978).

Varieties that are generated by their finite members

Observation

If a variety $\mathcal{V}$ is generated by its finite members, then $\mathbf{F}_{\mathcal{V}}(X)$ is a subdirect product of finite algebras.

Observation

If $\mathcal{V}$ is generated by its finite members, $\mathcal{V}$ is a variety of algebras with finitely many operation symbols, and $\mathcal{V}$ is defined by finitely many identities, then for every set X, the word problem for the free algebra $\mathbf{F}_{\mathcal{V}}(X)$ is solvable (Evans, Bull. Amer. Math. Soc., 1978).

Theorem (Freese, Transactions AMS, 1980)

The word problem for the free modular lattice over a five element set is unsolvable.

$\mathcal{N}$ is generated by its finite members

Theorem (EA, Monatshefte für Mathematik, 2004)

The variety of all near-rings is generated by its finite members. The variety of all zero-symmetric near-rings is generated by its finite members.

Idea of the proof

Every identity that fails in some near-ring even fails in some finite near-ring.

$\mathcal{N}$ is generated by its finite members

Theorem (EA, Monatshefte für Mathematik, 2004)

The variety of all near-rings is generated by its finite members. The variety of all zero-symmetric near-rings is generated by its finite members.

Idea of the proof

Every identity that fails in some near-ring even fails in some finite near-ring.

Task

Produce a finite near-ring in which

$$x_1 \circ (-x_2) \approx -(x_1 \circ x_2)$$

does not hold.

Take a prime p with $p > 2$, $\mathbf{N} := \mathbf{M}(\mathbb{Z}_p)$.

$$\begin{aligned} f(\gamma) &:= \gamma^2 && \text{for all } \gamma \in \mathbb{Z}_p \\ g(\gamma) &:= \gamma && \text{for all } \gamma \in \mathbb{Z}_p. \end{aligned}$$

Then $f \circ (-g)\ (\gamma) = (-\gamma)^2 = \gamma^2$, and $-(f \circ g)\ (\gamma) = -\gamma^2$.
Hence for $\eta_0 := 1$, we have

$$f \circ (-g)\ (\eta_0) \neq -(f \circ g)\ (\eta_0).$$

Task

Produce a finite near-ring in which

$$x_1 \circ (-x_2) \approx -(x_1 \circ x_2)$$

does not hold.

Take a prime p with $p > 2$, $\mathbf{N} := \mathbf{M}(\mathbb{Z}_p)$.

$$\begin{aligned} f(\gamma) &:= \gamma^2 && \text{for all } \gamma \in \mathbb{Z}_p \\ g(\gamma) &:= \gamma && \text{for all } \gamma \in \mathbb{Z}_p. \end{aligned}$$

Then $f \circ (-g)\ (\gamma) = (-\gamma)^2 = \gamma^2$, and $-(f \circ g)\ (\gamma) = -\gamma^2$.
Hence for $\eta_0 := 1$, we have

$$f \circ (-g)\ (\eta_0) \neq -(f \circ g)\ (\eta_0).$$

From near-rings to groups

Hence, instead of finding a finite model of

$$\text{Near-ring axioms} \cup \{x_1 \circ (-x_2) \not\approx -(x_1 \circ x_2)\},$$

we try to find a finite model of

$$\text{Group axioms} \cup \{\mathrm{f}_1(-\mathrm{f}_2(y_0)) \not\approx -\mathrm{f}_1(\mathrm{f}_2(y_0))\},$$

where f_i are unary function symbols.
So, the task is now:

> *Find a finite group* **G**, *an element* $\eta_0 \in G$, *and mappings* f_1, f_2 *on this group such that* $f_1(-f_2(\eta_0)) \neq -f_1(f_2(\eta_0))$.

Finding a finite model

Step 1 We translate

$$f_1(-f_2(y_0)) \not\approx -f_1(f_2(y_0)),$$

into

$$\begin{array}{lll}
s_0 = f_1(-f_2(y_0)), & t_0 = -f_1(f_2(y_0)), & B_0 = \emptyset, \\
s_1 = f_1(-y_1), & t_1 = -f_1(f_2(y_0)), & B_1 = \{y_1 \approx f_2(y_0)\}, \\
s_2 = f_1(y_2), & t_2 = -f_1(f_2(y_0)), & B_2 = B_1 \cup \{y_2 \approx -y_1\} \\
s_3 = y_3, & t_3 = -f_1(f_2(y_0)), & B_3 = B_2 \cup \{y_3 \approx f_1(y_2)\}, \\
s_4 = y_3, & t_4 = -f_1(y_4), & B_4 = B_3 \cup \{y_4 \approx f_2(y_0)\}, \\
s_5 = y_3, & t_5 = -y_5, & B_5 = B_4 \cup \{y_5 \approx f_1(y_4)\}, \\
s_6 = y_3, & t_6 = y_6, & B_6 = B_5 \cup \{y_6 \approx -y_5\}.
\end{array}$$

Thus, we are left with

$$\begin{array}{llllll}
y_3 & \not\approx & y_6 & & & \\
y_1 & \approx & f_2(y_0) & y_2 & \approx & -y_1 \\
y_3 & \approx & f_1(y_2) & y_4 & \approx & f_2(y_0) \\
y_5 & \approx & f_1(y_4) & y_6 & \approx & -y_5.
\end{array}$$

Step 2 We divide the formulas into E, F, G.

$$E := \{y_3 \not\approx y_6\}$$

$$F := \left\{ \begin{array}{ccc} y_1 & \approx & f_2(y_0) \\ y_3 & \approx & f_1(y_2) \\ y_4 & \approx & f_2(y_0) \\ y_5 & \approx & f_1(y_4) \end{array} \right\}$$

$$G := \left\{ \begin{array}{ccc} y_2 & \approx & -y_1 \\ y_6 & \approx & -y_5 \end{array} \right\}$$

Define $\sim$ on $\{0, 1, 2, \ldots, 6\}$ by

$$i \sim j \quad \text{iff} \quad \text{Group-axioms} \cup F \cup G \models y_i \approx y_j.$$

We have

$$1 \sim 4, \text{ all others are inequivalent.}$$

Finding models

We form

$$F' := \{y_i \approx y_j \mid i \sim j\}.$$

Now, for all i, j with $i \not\sim j$,

$$\text{Group-axioms} \cup F' \cup G \cup \{y_i \not\approx y_j\} \text{ is satisfiable.}$$

Since the variety of groups is generated by its finite members, we even find a **finite model** of

$$\text{Group-axioms} \cup F' \cup G \cup \{y_i \not\approx y_j\},$$

and hence of

$$\text{Group-axioms} \cup F' \cup G \cup \{y_r \not\approx y_s \mid r \not\sim s\}.$$

Now, we may define the interpretation of f_i such that F is satisfied.

Examples of composition algebras

- Let **A** be a set. Then $\langle M(A), \circ \rangle$ is a semigroup.
- Let $\langle G, +, -, 0 \rangle$ be a group. Then $\langle M(G), +, -, 0, \circ \rangle$ is a near-ring.
- Let $\langle R, +, -, 0, \cdot \rangle$ be a ring. Then $\langle M(R), +, -, 0, \cdot, \circ \rangle$ is a composition ring.

General version:

- Let $\langle A, F \rangle$ be an algebra in the variety $\mathcal{V}$. Then $\langle M(A), F \cup \{\circ\} \rangle$ is a $\mathcal{V}$-*composition algebra*.

Examples of composition algebras

- Let **A** be a set. Then $\langle M(A), \circ\rangle$ is a semigroup.
- Let $\langle G, +, -, 0\rangle$ be a group. Then $\langle M(G), +, -, 0, \circ\rangle$ is a near-ring.
- Let $\langle R, +, -, 0, \cdot\rangle$ be a ring. Then $\langle M(R), +, -, 0, \cdot, \circ\rangle$ is a composition ring.

General version:

- Let $\langle A, F\rangle$ be an algebra in the variety $\mathcal{V}$. Then $\langle M(A), F \cup \{\circ\}\rangle$ is a $\mathcal{V}$-*composition algebra*.

Examples of composition algebras

- Let **A** be a set. Then $\langle M(A), \circ \rangle$ is a semigroup.
- Let $\langle G, +, -, 0 \rangle$ be a group. Then $\langle M(G), +, -, 0, \circ \rangle$ is a near-ring.
- Let $\langle R, +, -, 0, \cdot \rangle$ be a ring. Then $\langle M(R), +, -, 0, \cdot, \circ \rangle$ is a composition ring.

General version:

- Let $\langle A, F \rangle$ be an algebra in the variety $\mathcal{V}$. Then $\langle M(A), F \cup \{\circ\} \rangle$ is a $\mathcal{V}$-*composition algebra*.

Examples of composition algebras

- Let **A** be a set. Then $\langle M(A), \circ \rangle$ is a semigroup.
- Let $\langle G, +, -, 0 \rangle$ be a group. Then $\langle M(G), +, -, 0, \circ \rangle$ is a near-ring.
- Let $\langle R, +, -, 0, \cdot \rangle$ be a ring. Then $\langle M(R), +, -, 0, \cdot, \circ \rangle$ is a composition ring.

General version:

- Let $\langle A, F \rangle$ be an algebra in the variety $\mathcal{V}$. Then $\langle M(A), F \cup \{\circ\} \rangle$ is a $\mathcal{V}$*-composition algebra*.

The definition of composition algebras

Let $\mathcal{L}$ be a language of algebras. We consider algebras of language $\mathcal{C}(\mathcal{L})$, where $\mathcal{C}(\mathcal{L})$ contains all function symbols of $\mathcal{L}$ plus one new binary symbol $\circ$ added.

Definition

Let $\mathcal{L}$ be a language of algebras, and let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. A $\mathcal{C}(\mathcal{L})$-algebra **N** is called a $\mathcal{K}$-*composition algebra* iff the $\mathcal{L}$-reduct of **N** lies in $\mathcal{K}$, and **N** satisfies the identities

$$(x_1 \circ x_2) \circ x_3 \approx x_1 \circ (x_2 \circ x_3)$$

and

$$\omega(x_1, \ldots, x_k) \circ x_{k+1} \approx \omega(x_1 \circ x_{k+1}, \ldots, x_k \circ x_{k+1}),$$

where $k \in \mathbb{N} \cup \{0\}$ and ω is a k-ary function symbol of $\mathcal{L}$.

The definition of composition algebras

Let $\mathcal{L}$ be a language of algebras. We consider algebras of language $\mathcal{C}(\mathcal{L})$, where $\mathcal{C}(\mathcal{L})$ contains all function symbols of $\mathcal{L}$ plus one new binary symbol $\circ$ added.

Definition

Let $\mathcal{L}$ be a language of algebras, and let $\mathcal{K}$ be a class of $\mathcal{L}$-algebras. A $\mathcal{C}(\mathcal{L})$-algebra **N** is called a $\mathcal{K}$*-composition algebra* iff the $\mathcal{L}$-reduct of **N** lies in $\mathcal{K}$, and **N** satisfies the identities

$$(x_1 \circ x_2) \circ x_3 \approx x_1 \circ (x_2 \circ x_3)$$

and

$$\omega(x_1, \ldots, x_k) \circ x_{k+1} \approx \omega(x_1 \circ x_{k+1}, \ldots, x_k \circ x_{k+1}),$$

where $k \in \mathbb{N} \cup \{0\}$ and ω is a k-ary function symbol of $\mathcal{L}$.

The class of composition algebras

$\mathbb{C}(\mathcal{K})$:= the class of all $\mathcal{K}$-composition algebras.
For every algebra $\mathbf{A}$ in a variety $\mathcal{V}$, the full function algebra $\mathbf{M}(\mathbf{A})$ lies in $\mathbb{C}(\mathcal{V})$.

Definition

Let $\mathcal{V}$ be a variety of $\mathcal{L}$-algebras, and let $\mathcal{F}$ be a subclass of $\mathcal{V}$. Then we define the class $\mathbb{M}(\mathcal{F})$ as the subclass of $\mathbb{C}(\mathcal{V})$ given by

$$\mathbb{M}(\mathcal{F}) := \{\mathbf{M}(\mathbf{A}) \,|\, \mathbf{A} \in \mathcal{F}\}.$$

What we really have proved

Theorem

Let $\mathcal{L}$ be a language of algebras, let $\mathcal{F}$ be a class of $\mathcal{L}$-algebras, and let $\mathcal{V} := \mathbb{HSP}(\mathcal{F})$. Then the variety of $\mathcal{V}$-composition algebras is generated by the class of all $\mathbf{M}(\mathbf{A})$, where $\mathbf{A} \in \mathbb{P}_{\text{fin}}(\mathcal{F})$. In other words, we have

$$\mathbb{C}(\mathcal{V}) = \mathbb{HSP}(\mathbb{M}(\mathbb{P}_{\text{fin}}(\mathcal{F}))).$$

Brief summary:

$$\mathbb{C}(\mathbb{HSP}(\mathcal{F})) = \mathbb{HSP}(\mathbb{M}(\mathbb{P}_{\text{fin}}(\mathcal{F}))).$$

Consequences

Corollary

Let p be a prime. Then the variety of near-rings is generated by $\{\mathbf{M}(\mathbf{G}) \mid \mathbf{G}$ is a finite p-group$\}$.

Let $\mathcal{V}$ be a variety of algebras such that $\mathcal{V}$ is generated by its finite members. Then the variety $\mathbb{C}(\mathcal{V})$ is also generated by its finite members.

Hence, also the variety $\mathcal{A}$ of near-rings with abelian addition is generated by its finite members, and hence has a solvable word-problem.

Consequences

Corollary

Let p be a prime. Then the variety of near-rings is generated by $\{\mathbf{M}(\mathbf{G}) \mid \mathbf{G}$ is a finite p-group$\}$.

Let $\mathcal{V}$ be a variety of algebras such that $\mathcal{V}$ is generated by its finite members. Then the variety $\mathbb{C}(\mathcal{V})$ is also generated by its finite members.

Hence, also the variety $\mathcal{A}$ of near-rings with abelian addition is generated by its finite members, and hence has a solvable word-problem.

Consequences

Corollary

Let p be a prime. Then the variety of near-rings is generated by $\{\mathbf{M}(\mathbf{G}) \mid \mathbf{G}$ is a finite p-group$\}$.

Let $\mathcal{V}$ be a variety of algebras such that $\mathcal{V}$ is generated by its finite members. Then the variety $\mathbb{C}(\mathcal{V})$ is also generated by its finite members.

Hence, also the variety $\mathcal{A}$ of near-rings with abelian addition is generated by its finite members, and hence has a solvable word-problem.