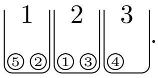
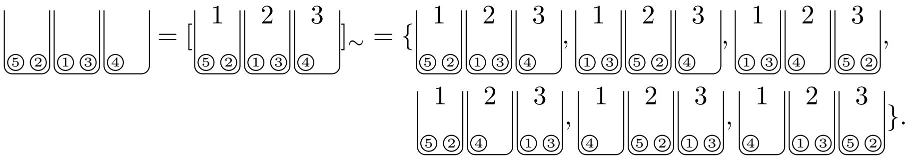
1 Recall: counting of functions modulo group actions

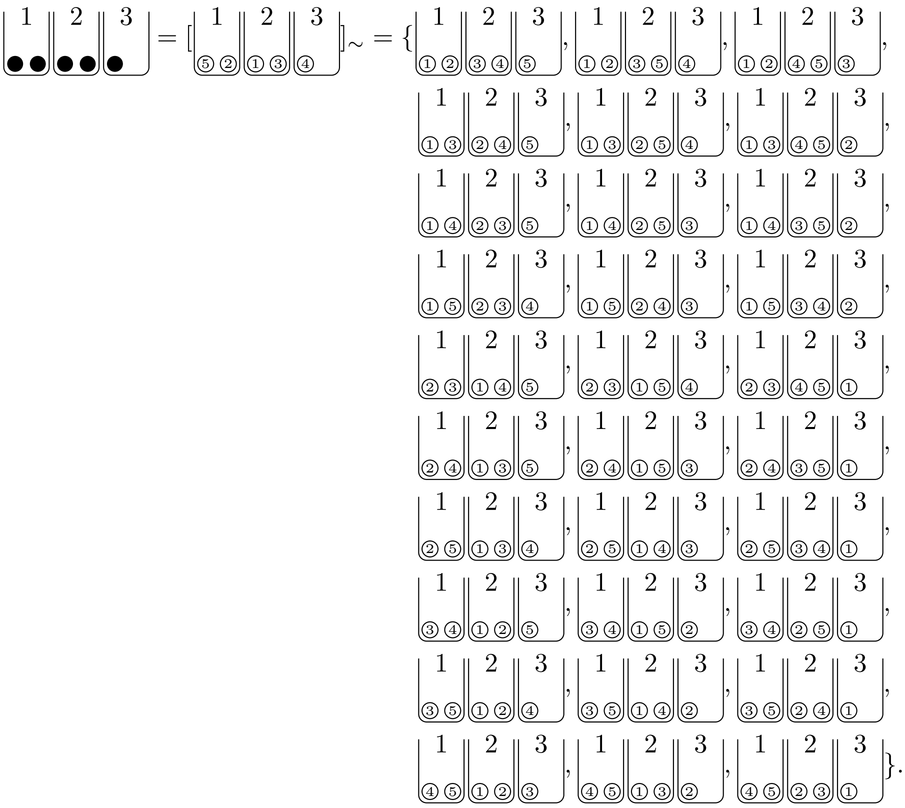
- A function $f\colon \{1,\dots,n\} \to \{1,\dots,k\}$ is an assignment of n distinguishable balls into k distinguishable pots. Writing $N = \{1,\dots,n\}$, $K = \{1,\dots,k\}$, the set of all functions $N \to K$ is denoted by K^N. We have $|K^N| = k^n$.

  Example: 

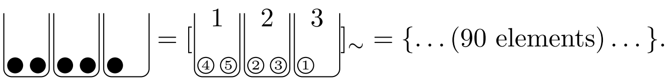
- To model the case of indistinguishable pots, declare that $f \sim g \iff \exists\, \pi \in S_k : f = \pi \circ g$. Then the assignments are exactly the equivalence classes, viz. the orbits of the group action.

  Example: 

- Similarly, indistinguishable balls are modeled via $f \sim g \iff \exists\, \sigma \in S_n : f = g \circ \sigma^{-1}$. Again, the desired assignments are exactly the equivalence classes, viz. the orbits of the group action.

  Example: 

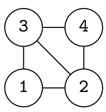
- Finally, if balls as well as pots are indistinguishable, we can use $f \sim g \iff \exists\, \pi \in S_n, \sigma \in S_k : f = \pi \circ g \circ \sigma^{-1}$.

  Example: 

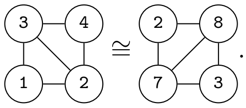
- For each of these cases, we may ask how many functions are there, and how many of them are injective, surjective, bijective. The corresponding formulas are listed in a table in the lecture notes.

2 Recall: automorphisms of graphs

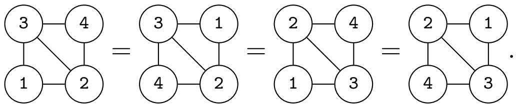
- A graph is a pair $G = (V, E)$ where V is a finite set and E is a subset of $V \times V$. The elements of V are called *vertices* of the graph, and the elements of E are called *edges*.

  Example: 

- Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are *isomorphic* (to each other) iff there exists a bijective function $h\colon V_1 \to V_2$ such that for all $u, v \in V_1$ we have $(u, v) \in E_1 \iff (h(u), h(v)) \in E_2$. Such a function is called an isomorphism for G_1, G_2.

  Example: 

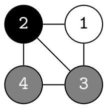
- Each graph G is isomorphic to itself, as the identity function is an isomorphism. There may be further isomorphisms. An isomorphism from a graph to itself is called an automorphism. The automorphisms of a graph form a group with composition.

  Example: 

  $\mathrm{Aut}(G) = \{\mathrm{id}, (1\ 4), (2\ 3), (1\ 4)(2\ 3)\} \subseteq S_4$.

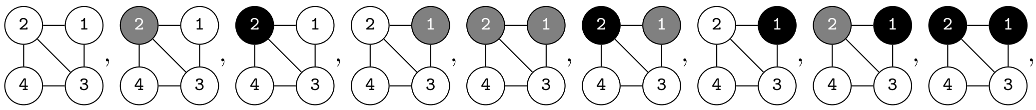
3 A combination of both concepts: Graph Colorings

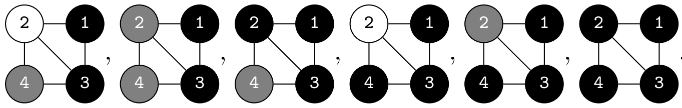
- Now let $G = (V, E)$ be a graph and let $K = \{1, \ldots, k\}$ be a set. A graph coloring is a function $f\colon V \to K$. It assigns to every vertex $v \in V$ a "color" $f(v) \in K$.

  Example: 

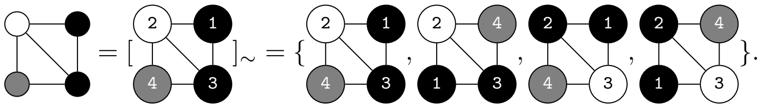
1	2	3	4
w	b	g	g

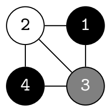
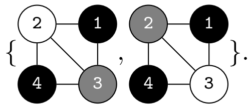
- For a given graph and a given K, we may wonder how many different colorings there are. Of course, the answer is $k^{|V|}$.

  Example: 

  ....,  (Altogether 81 colorings)

- If we only care about the structure but not about the names of the vertices, we can identify assignments that can be transformed into one another by a graph automorphism: $f \sim g \iff \exists\ \pi \in \mathrm{Aut}(G) : f = g \circ \pi^{-1}$.

  Example: 

  Note that for example  does not belong to the same equivalence class, although it is also a coloring with one white, one gray, and two black vertices. This coloring's class is . Note that this class has only two elements.

  For this example graph, there are altogether 36 graph colorings up to automorphisms: we can consider independently the coloring of 1, 4 and the coloring of 2, 3, and in each case, the coloring corresponds to an assignment of two indistinguishable balls (the vertices) to three distinguishable pots (the colors). As there are six such assignments, we get 36 combinations of two such assignments. Here they are: