Dominance in the family of Sugeno-Weber t-norms

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Abstract

The dominance relationship between two members of the family of Sugeno Weber t-norms is proven by using a quantifer elimination algorithm. Further it is shown that dominance is a transitive, and therefore also an order relation, on this family of t-norms.

1. Introduction

Dominance is a functional inequality which arises in different application fields. It most often appears when discussing the preservation of properties during (dis-)aggregation processes like, e.g., in flexible querying, preference modelling or computer-assisted assessment [5, 7, 20, 23]. It is further crucial in the construction of Cartesian products of probabilistic metric and normed spaces [8, 31, 36] as well as when constructing many-valued equivalence and order relations [2, 5, 6, 39].

Introduced in 1976 in the framework of probabilistic metric spaces as an inequality involving two triangle functions (see Tardiff [36] and Schweizer and Sklar [31] for an early generalization to operations on a partially ordered set), it was soon clear that dominance constitutes a reflexive and antisymmetric relation on the set of all t-norms. That it is not a transitive relation has been proven much later in 2007 [30]. This negative answer to a long open question has, to some extent, been surprising. In particular since earlier results showed that for several important single-parametric families of t-norms, dominance is also a transitive and therefore an order relation [10, 21, 24, 25, 29, 33].

The family of Sugeno-Weber t-norms has been one of the more prominent families of t-norms for which dominance has not been completely characterized so far. First partial results were obtained recently [22] by invoking results on different sufficient conditions derived from a generalization of the Mulholland inequality [26] and involving the additive generators of the t-norms, their pseudo-inverses and their derivatives [22].

The purpose of this paper is to close this gap. We present a proof for a complete characterization of dominance in the family of Sugeno-Weber t-norms. This is interesting because from all the families of t-norms discussed in Section 4 in the monograph of Klement, Mesiar and Pap [10], the family of Sugeno-Weber t-norms was, until now, the only family for which no complete classification result was available. But there are further aspects which make our results interesting:

First, the solution set is of a completely different form than witnessed before for other families. So far dominance in single-parametric families has either been in complete accordance with the ordering in the family (i.e., dominance

constitutes a linear order on the family of t-norms) or dominance has rarely appeared between family members, i.e., holds only in the trivial cases of self-dominance or when involving maximal or minimal elements of the family. For the family of Sugeno-Weber t-norms neither is the case.

Second, although the solution set looks different, dominance is a transitive relation on the family of Sugeno-Weber t-norms.

Third, the results have been achieved by the use of symbolic computation algorithms. More explicitly a quantifier elimination algorithm for real closed fields (Cylindrical Algebraic Decomposition) has, after several transformation steps, been applied to logical equivalent formulations of the original problems. The present contribution is therefore also an example of a successful application of computer algebra and symbolic computation for solving polynomial inequalities.

The following preliminaries shall clarify the necessary notions and summarize basic facts about the family of Sugeno-Weber t-norms. Dominance as well as some basic aspects of quantifier elimination algorithms will be explained. Then we provide and prove the main results — the characterization of dominance between two t-norms of the family of Sugeno-Weber t-norms and transitivity of dominance in the family. We finally discuss the results in more detail.

2. Preliminaries

2.1. Triangular norms

We briefly summarize some basic properties of t-norms for a thorough understanding of this paper. Excellent overviews on and discussions of triangular norms (including historical accounts, further details, proofs, and references) can be found in the literature [1, 9, 10, 11, 12, 13].

Definition 1. *A* triangular norm *(briefly* t-norm*)* $T\colon [0,1]^2 \to [0,1]$ *is a binary operation on the unit interval which is commutative, associative, non-decreasing and has neutral element* 1.

The most prominent examples of t-norms are the *minimum* $T_{\mathbf{M}}$, the *product* $T_{\mathbf{P}}$, the *Łukasiewicz t-norm* $T_{\mathbf{L}}$ and the *drastic product* $T_{\mathbf{D}}$. They are defined by $T_{\mathbf{M}}(u,v) = \min(u,v)$, $T_{\mathbf{P}}(u,v) = u \cdot v$, $T_{\mathbf{L}}(u,v) = \max(u+v-1,0)$, and

$$T_{\mathbf{D}}(u,v) = \begin{cases} \min(u,v), & \text{if } \max(u,v) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the basic t-norms $T_{\mathbf{M}}, T_{\mathbf{P}}$ and $T_{\mathbf{L}}$ are continuous, whereas the drastic product $T_{\mathbf{D}}$ is not. The comparison of two t-norms is done pointwisely, i.e., if, for all $x, y \in [0,1]$, it holds that $T_1(x,y) \geq T_2(x,y)$, then we say that T_1 is *stronger* than T_2 and denote it by $T_1 \geq T_2$. The minimum $T_{\mathbf{M}}$ is the strongest of all t-norms, the drastic product $T_{\mathbf{D}}$ is the weakest of all t-norms. Moreover, the four basic t-norms are ordered in the following way: $T_{\mathbf{D}} \leq T_{\mathbf{L}} \leq T_{\mathbf{P}} \leq T_{\mathbf{M}}$.

Definition 2. *A t-norm T is called*

(i) Archimedean *if for all $u, v \in]0,1[$ there exists an $n \in \mathbb{N}$, $n \geq 2$, such that*

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) < v \quad \text{where} \quad T(\underbrace{u,\ldots,u}_{n \text{ times}}) = T(u, T(\underbrace{u,\ldots,u}_{n-1 \text{ times}})) \text{ for all } n \geq 3.$$

(ii) A t-norm T is called strict *if it is continuous and strictly monotone, i.e., for all $u, v, w \in [0,1]$ it holds that*

$$T(u,v) < T(u,w) \quad \text{whenever} \quad u > 0 \text{ and } v < w\,.$$

(iii) A t-norm T is called nilpotent *if it is continuous and if each $u \in]0,1[$ is a nilpotent element of T, i.e., there exists some $n \in \mathbb{N}$ such that*

$$T(\underbrace{u,\ldots,u}_{n \text{ times}}) = 0\,.$$

Note that for a strict t-norm T it holds that $T(u,v) > 0$ for all $u, v \in]0,1]$, while for a nilpotent t-norm T it holds that for every $u \in]0,1[$ there exists some $v \in]0,1[$ such that $T(u,v) = 0$ (each $u \in]0,1[$ is a so-called *zero divisor*). Therefore for a nilpotent t-norm T_1 and a strict t-norm T_2 it can never hold that $T_1 \geq T_2$.

2.1.1. The family of Sugeno-Weber t-norms

In 1983 S. Weber proposed the use of this particular family for modelling the intersection of fuzzy sets [40]. Since then, its dual operations, the Sugeno-Weber t-conorms, defined for all $\lambda \in [0,\infty]$ and all $u,v \in [0,1]$ by $S_\lambda^{\mathbf{SW}}(u,v) = 1-T_\lambda^{\mathbf{SW}}(1-u,1-v)$, have played a prominent role for generalized decomposable measures [14, 15, 19, 41], in particular, since they already appeared as possible generalized additions in the context of λ-fuzzy measures [35].

The family of Sugeno-Weber t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$ is, for all $u,v \in [0,1]$, given by

$$T_\lambda^{\mathbf{SW}}(u,v) = \begin{cases} T_{\mathbf{P}}(u,v), & \text{if } \lambda = 0, \\ T_{\mathbf{D}}(u,v), & \text{if } \lambda = \infty, \\ \max(0,(1-\lambda)uv + \lambda(u+v-1)), & \text{if } \lambda \in]0,\infty[\,. \end{cases}$$

The family is of particular interest since all but two of its members are nilpotent t-norms. Parameters $\lambda \in]0,\infty[$ lead to nilpotent t-norms (with $T_1^{\mathbf{SW}} = T_{\mathbf{L}}$ as special case), while $T_0^{\mathbf{SW}} = T_{\mathbf{P}}$ is the only strict member. For $\lambda \in [0,\infty[$, the Sugeno-Weber t-norms are continuous Archimedean t-norms [16, 35, 40], for $\lambda \in [0,1]$ they are also copulas [18].

For $\lambda \in]0,1[$, an interesting characterization of the family members has been provided by Mayor [16]: A t-norm T is a Sugeno-Weber t-norm $T_\lambda^{\mathbf{SW}}$ with $\lambda \in]0,1[$ if and only if T is nilpotent and for each $w \in [0,1]$, the graph of the vertical section $T(w,.)$ of T is a straight line segment on $\left[\frac{\lambda(1-w)}{\lambda(1-w)+w}, 1\right]$.

2.2. Dominance

The dominance relation has, as t-norms do, its roots in the field of probabilistic metric spaces [31, 36]. It was originally introduced for associative operations (with common neutral element) on a partially ordered set [31], and has been further investigated for t-norms [21, 29, 30, 37] and aggregation functions [17, 20, 23]. For more recent results on dominance between triangle functions resp. operations on distance distribution functions see also the article by Saminger-Platz and Sempi [27].

We state the definition for t-norms only.

Definition 3. *Consider two t-norms T_1 and T_2. We say that T_1* dominates *T_2 (or T_2 is dominated by T_1), denoted by $T_1 \gg T_2$, if, for all $x,y,u,v \in [0,1]$, it holds that*

$$T_1(T_2(x,y),T_2(u,v)) \geq T_2(T_1(x,u),T_1(y,v))\,. \tag{1}$$

As mentioned already earlier, the dominance relation, in particular between t-norms, plays an important role in various topics, such as the construction of Cartesian products of probabilistic metric and normed spaces [8, 31, 36], the construction of many-valued equivalence relations [5, 6, 39] and many-valued order relations [2], the preservation of various properties during (dis-)aggregation processes in flexible querying, preference modelling and computer-assisted assessment [5, 7, 20, 23].

Every t-norm, in fact every function non-decreasing in each of its arguments, is dominated by $T_{\mathbf{M}}$. Moreover, every t-norm dominates itself and $T_{\mathbf{D}}$. Since all t-norms have neutral element 1, dominance between two t-norms implies their comparability: $T_1 \gg T_2$ implies $T_1 \geq T_2$. The converse does not hold.

Due to the induced comparability it also follows that dominance is an antisymmetric relation on the class of t-norms. Associativity and symmetry ensure that dominance is also reflexive on the class of t-norms.

Although dominance is not a transitive relation on the set of continuous, and therefore also not on the set of all, t-norms (see the results by Sarkoci [24, 28, 30]), it is transitive on several single-parameteric families of t-norms (see also Table 1).

It is interesting to see that in all the cases displayed in Table 1 dominance is either in complete accordance with the ordering in the family (i.e., dominance constitutes a linear order on the family of t-norms) or dominance rarely appears among family members, i.e., holds only in the trivial cases of self-dominance and dominance involving maximal or minimal elements of the family (for an overview on these results and referential details we refer to the article by Saminger-Platz [25]).

We will show below that dominance between two members of the family of Sugeno-Weber t-norms is of a completely different type, although finally dominance also turns out to be a transitive relation on this family.

Family of t-norms	$T_\lambda \gg T_\mu$	Hasse-Diagrams
Schweizer-Sklar $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$ (Sherwood, 1984)	$\lambda \le \mu$	
Aczél-Alsina $(T_\lambda^{\mathbf{AA}})_{\lambda\in[0,\infty]}$ Dombi $(T_\lambda^{\mathbf{D}})_{\lambda\in[0,\infty]}$ Yager $(T_\lambda^{\mathbf{Y}})_{\lambda\in[0,\infty]}$ (Klement, Mesiar, Pap, 2000)	$\lambda \ge \mu$	
$(T_\lambda^{\mathbf{8}})_{\lambda\in[0,\infty]}$ $(T_\lambda^{\mathbf{15}})_{\lambda\in[0,\infty]}$ $(T_\lambda^{\mathbf{22}})_{\lambda\in[0,\infty]}$ $(T_\lambda^{\mathbf{23}})_{\lambda\in[0,\infty]}$ (Saminger-Platz, 2009)	$\lambda \ge \mu$	
Frank $(T_\lambda^{\mathbf{F}})_{\lambda\in[0,\infty]}$ Hamacher $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,\infty]}$ (Sarkoci, 2005)	$\lambda = 0, \lambda = \mu, \mu = \infty$	
$(T_\lambda^{\mathbf{9}})_{\lambda\in[0,\infty]}$ (Saminger-Platz, 2009)	$\lambda = \infty, \lambda = \mu, \mu = 0$	
Mayor-Torrens $(T_\lambda^{\mathbf{MT}})_{\lambda\in[0,1]}$ Dubois-Prade $(T_\lambda^{\mathbf{DP}})_{\lambda\in[0,1]}$ (Saminger, De Baets, De Meyer, 2005)	$\lambda = 0, \lambda = \mu$	

Table 1: Dominance relation in several families of t-norms

2.2.1. The family of Sugeno-Weber t-norms

The members of the family form a decreasing sequence of t-norms with respect to their parameter, i.e., $T_\lambda^{\mathbf{SW}} \ge T_\mu^{\mathbf{SW}}$ if and only if $\lambda \le \mu$. Since dominance induces order on the t-norms involved, it is therefore clear that a necessary condition for $T_\lambda^{\mathbf{SW}} \gg T_\mu^{\mathbf{SW}}$ is that $\lambda \le \mu$.

Dominance among the family members has further been studied by invoking results on different sufficient conditions derived from a generalization of the Mulholland inequality [26] and involving the additive generators of the t-norms, their pseudo-inverses and their derivatives [22]. The results obtained did not lead to a full characterization of dominance in the whole family, but already indicated that the dominance relationship might be of a completely different structure than the dominance relationship laid bare in any other family before. We quote the following result.

Proposition 1. *[22] Consider the family of Sugeno-Weber t-norms* $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$. *For all* $\lambda, \mu \in [0,\infty]$ *such that one of the following condition holds*

(i) $\lambda \le \min(1, \mu)$,

(ii) $1 < \lambda \le \mu \le r^*$, *with* $r^* = 6.00914$ *denoting the second root of* $\log^2(t) + \log(t) - t + 1$,

it follows that $T_\lambda^{\mathbf{SW}} \gg T_\mu^{\mathbf{SW}}$.

2.3. Quantifier Elimination and Cylindrical Algebraic Decomposition

In contrast to many other families of t-norms, the dominance relation for Sugeno-Weber t-norms does not involve any logarithms or exponentials but can be formulated by addition, multiplication and the max-operation only. This is a striking structural advantage, because there are algorithms available for proving this kind of formulas automatically. The first algorithm for proving formulas about polynomial inequalities was already given by Tarski in the

early 1950s [38] but his algorithm was only of theoretical interest. Nowadays, modern implementations [3, 32, 34] of Collins' algorithm for Cylindrical Algebraic Decomposition (CAD) [4] make it possible to actually perform nontrivial computations within a reasonable amount of time. They are meanwhile established as valuable tools for solving problems about polynomial inequalities.

In general, the input to CAD is a formula of the form

$$Q_1\, x_1 \in \mathbb{R} \;\cdots\; Q_n\, x_n \in \mathbb{R} : A(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

where the Q_i are quantifiers (either $\forall$ or $\exists$) and A is a boolean combination of polynomial equations and inequalities in the variables x_i and y_i. The variables x_i are bound by quantifiers, the variables y_i are free. Given such a formula, the algorithm computes a quantifier free formula $B(y_1, \ldots, y_m)$ which is equivalent to the input formula.

A simple example is given by

$$\forall\, x \in \mathbb{R}\; \exists\, y \in \mathbb{R} : (x-1)(y-1) > 1 \Leftrightarrow x^2 + y^2 - z^2 > 0,$$

where $A(x, y, z)$ is the formula $(x-1)(y-1) > 1 \Leftrightarrow x^2 + y^2 - z^2 > 0$, the bound variables are x, y, and z is a free variable. Applied to this formula, the CAD algorithm may return the quantifier free formula $B(z) = z \leq -1 \;\vee\; z \geq 1$. This formula is equivalent to the quantified formula in the sense that for every real number $z \in \mathbb{R}$ the input formula holds if and only if the output formula holds.

Applied to a quantified formula with no free variables, CAD will return one of the two logical constants True or False. Applied to a formula with only free variables, CAD will produce an equivalent formula in the same variables which is normalized in a certain sense.

The basic idea of the CAD algorithm is as follows: in a first step the algorithm computes a decomposition of $\mathbb{R}^{n+m}$ into a finite (but potentially huge) number of regions with the property that the quantifier free part $A(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of the quantified formula is true at a point $P = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ if and only if it is true for all points in the region containing P. In the second step it performs some operations on these finitely many regions eliminating variables bound by quantifiers. This gives in the end a decomposition of $\mathbb{R}^m$ into finitely many regions with the property that the whole quantified formula is true at a point $P = (y_1, \ldots, y_m)$ if and only if it is true for every point in the region containing P. In the final step the algorithm selects those regions where the quantified formula is true and returns a formal description for the union of these regions as output. For details we refer once more to the literature [4].

It must be stressed that the equivalence of input and output of CAD is not approximate in any way but completely rigorous. In particular, if CAD applied to a certain formula Φ yields the output True, then the trace of the computation constitutes a lengthy and ugly and insightless but correct and complete and checkable proof of Φ. The price to be paid for such a strictly correct output is that computations may take very long. While it is guaranteed that every CAD computation will eventually terminate and produce a correct output, such a guarantee is of little use if the expected runtime exceeds by far our expected lifetime.

In its original formulation, the dominance relation for Sugeno-Weber t-norms is an example for a formula which CAD can do in principle but not in practice. Human interaction is necessary to break the big computation into several smaller ones, to properly reformulate intermediate results, and to exploit common properties of different parts of the problem. We succeeded in finding and formulating such necessary and appropriate intermediate steps finally leading to a successful and computable proof that is explained on five pages, expanded in the Mathematica file available at http://www.risc.jku.at/people/mkauers/sugeno-weber/proof.nb and executed in 15 to 30 minutes depending on the actual computation capacities and the Mathematica version used.

3. Main results

Theorem 2. *Consider the family of Sugeno-Weber t-norms $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$. Then, for all $\lambda, \mu \in [0, \infty]$, $T_\lambda^{\mathbf{SW}}$ dominates $T_\mu^{\mathbf{SW}}$, $T_\lambda^{\mathbf{SW}} \gg T_\mu^{\mathbf{SW}}$, if and only if one of the following conditions holds:*

(i) $\lambda = 0$,

(ii) $\mu = \infty$,

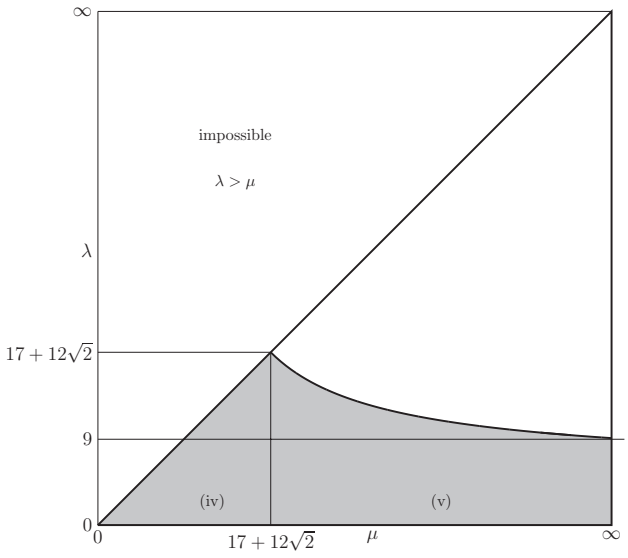

Figure 1: Relationship between parameters λ and μ for $T_\lambda^{\mathbf{SW}}$ dominating $T_\mu^{\mathbf{SW}}$

(iii) $\lambda = \mu$,

(iv) $0 < \lambda < \mu \leq 17 + 12\sqrt{2}$,

(v) $17 + 12\sqrt{2} < \mu$ *and* $0 < \lambda \leq \left(\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}}\right)^2$.

In Fig. 1, we have illustrated for which parameters λ for a given parameter μ it holds that $T_\lambda^{\mathbf{SW}}$ dominates $T_\mu^{\mathbf{SW}}$. In Section 4 the solution sets will be discussed in more detail.

Based on the characterization of dominance between two members of the family of Sugeno-Weber t-norms, another CAD computation directly applied to the problem of transitivity in the family asserts the transitivity of the relation on the family within less than two seconds. Note also that an alternative proof of the transitivity of dominance in the family of Sugeno-Weber t-norms is given in Section 4.3. In any case, we can state:

Proposition 3. *Dominance is a transitive, and therefore an order relation on the set of all Sugeno-Weber t-norms.*

The remainder of this section is devoted to the proof of Theorem 2, to the necessary steps of reformulating intermediate proof steps, exploiting properties of subparts of the problem, and finding an (equivalent) formulation of the original problem computable and solvable by CAD in reasonable time.

Consider some $\lambda, \mu \in [0, \infty]$. In case that $\lambda = 0$, $\mu = \infty$, or $\lambda = \mu$ the result is trivial since $T_{\mathbf{M}} = T_0^{\mathbf{SW}}$ dominates all, $T_{\mathbf{D}} = T_\infty^{\mathbf{SW}}$ is dominated by all t-norms, and every t-norm dominates itself. Since dominance induces order, we may assume without loss of generality that $0 < \lambda < \mu < \infty$. Moreover, the dominance inequality

$$T_\lambda^{\mathbf{SW}}(T_\mu^{\mathbf{SW}}(x,y), T_\mu^{\mathbf{SW}}(u,v)) \geq T_\mu^{\mathbf{SW}}(T_\lambda^{\mathbf{SW}}(x,u), T_\lambda^{\mathbf{SW}}(y,v))$$

is trivially fulfilled whenever $1 \in \{x, y, u, v\}$ or $0 \in \{x, y, u, v\}$. Therefore it suffices to solve the following problem:

Determine all $\lambda, \mu \in]0, \infty[$ with $\lambda < \mu$ such that for all $x, y, u, v \in]0, 1[$:

$$T_\lambda^{\mathbf{SW}}(T_\mu^{\mathbf{SW}}(x,y), T_\mu^{\mathbf{SW}}(u,v)) \geq T_\mu^{\mathbf{SW}}(T_\lambda^{\mathbf{SW}}(x,u), T_\lambda^{\mathbf{SW}}(y,v))$$

or explicitly such that

$$
\begin{aligned}
&\forall\, x,y,u,v \in\,]0,1[\,: \\
&\quad \max\Big(0,(1-\lambda)\max(0,(1-\mu)uv+\mu(u+v-1))\max(0,(1-\mu)xy+\mu(x+y-1)) \\
&\qquad +\lambda(\max(0,(1-\mu)uv+\mu(u+v-1))+\max(0,(1-\mu)xy+\mu(x+y-1))-1)\Big) \\
&\quad \geq \max\Big(0,(1-\mu)\max(0,(1-\lambda)ux+\lambda(u+x-1))\max(0,(1-\lambda)vy+\lambda(v+y-1)) \\
&\qquad +\mu(\max(0,(1-\lambda)ux+\lambda(u+x-1))+\max(0,(1-\lambda)vy+\lambda(v+y-1))-1)\Big).
\end{aligned}
$$

This problem can in principle be solved directly by a quantifier elimination algorithm (like, e.g., CAD) for real closed fields. However, in practice this computation would take very long. By a series of appropriate simplifications we can reduce the computation time tremendously. The following simplification steps result in an equivalent quantified formula for which quantifier elimination takes a few minutes only.

1. *Eliminate the outer maxima.* The body of the formula in question has the form $\max(0,A) \geq \max(0,B)$. It is readily confirmed by hand, or by CAD, that

$$\max(0,A) \geq \max(0,B) \iff B \leq 0 \ \vee\ A \geq B \iff B \leq 0 \ \vee\ A \geq B > 0$$

for all real A, B. Applying the last equivalence and making the range restrictions on x,y,u,v and λ,μ explicit, we arrive at the equivalent formulation

$$
\begin{aligned}
&\forall\, x,y,u,v \in \mathbb{R}: \Big((0<\lambda<\mu<\infty)\ \wedge\ (0<x<1)\ \wedge\ (0<y<1)\ \wedge\ (0<u<1)\ \wedge\ (0<v<1)\Big) \\
&\Rightarrow \Big(\Big((1-\mu)\max(0,(1-\lambda)ux+\lambda(u+x-1))\max(0,(1-\lambda)vy+\lambda(v+y-1)) \\
&\qquad +\mu(\max(0,(1-\lambda)ux+\lambda(u+x-1))+\max(0,(1-\lambda)vy+\lambda(v+y-1))-1)\leq 0\Big) \\
&\quad \vee\ \Big((1-\lambda)\max(0,(1-\mu)uv+\mu(u+v-1))\max(0,(1-\mu)xy+\mu(x+y-1)) \\
&\qquad +\lambda(\max(0,(1-\mu)uv+\mu(u+v-1))+\max(0,(1-\mu)xy+\mu(x+y-1))-1)) \\
&\qquad \geq (1-\mu)\max(0,(1-\lambda)ux+\lambda(u+x-1))\max(0,(1-\lambda)vy+\lambda(v+y-1)) \\
&\qquad +\mu(\max(0,(1-\lambda)ux+\lambda(u+x-1))+\max(0,(1-\lambda)vy+\lambda(v+y-1))-1)>0\Big)\Big)
\end{aligned}
$$

2. *Eliminate the inner maxima.* The new formula still contains four different maximum expressions:

$$
\begin{aligned}
&\max(0,(1-\mu)uv+\mu(u+v-1)) \text{ and } \max(0,(1-\mu)xy+\mu(x+y-1)) \text{ in } A; \\
&\max(0,(1-\lambda)ux+\lambda(u+x-1)) \text{ and } \max(0,(1-\lambda)vy+\lambda(v+y-1)) \text{ in } B.
\end{aligned}
$$

To get rid of these, observe that if $\Phi(X)$ is a formula depending on a real variable X, then the following equivalences are valid

$$
\begin{aligned}
\Phi(\max(0,X)) &\iff (X \leq 0 \ \vee\ X > 0)\ \wedge\ \Phi(\max(0,X)) \\
&\iff (X \leq 0\ \wedge\ \Phi(0))\ \vee\ (X > 0\ \wedge\ \Phi(X)).
\end{aligned}
$$

For a formula depending on several real variables, this rewriting yields

$$
\begin{aligned}
&\Phi(\max(0,X_1),\max(0,X_2),\max(0,X_3),\max(0,X_4)) \\
&\iff (X_1 \leq 0\ \wedge\ X_2 \leq 0\ \wedge\ X_3 \leq 0\ \wedge\ X_4 \leq 0\ \wedge\ \Phi(0,0,0,0) \\
&\quad \vee\ X_1 > 0\ \wedge\ X_2 \leq 0\ \wedge\ X_3 \leq 0\ \wedge\ X_4 \leq 0\ \wedge\ \Phi(X_1,0,0,0) \\
&\quad \vee\ X_1 \leq 0\ \wedge\ X_2 > 0\ \wedge\ X_3 \leq 0\ \wedge\ X_4 \leq 0\ \wedge\ \Phi(0,X_2,0,0) \\
&\quad \vee\ X_1 > 0\ \wedge\ X_2 > 0\ \wedge\ X_3 \leq 0\ \wedge\ X_4 \leq 0\ \wedge\ \Phi(X_1,X_2,0,0) \\
&\quad \vdots \\
&\quad \vee\ X_1 > 0\ \wedge\ X_2 > 0\ \wedge\ X_3 > 0\ \wedge\ X_4 > 0\ \wedge\ \Phi(X_1,X_2,X_3,X_4)).
\end{aligned}
$$

Applying these considerations to our problem, we first put

$$\begin{aligned} X_1 &= (1-\lambda)ux + \lambda(u+x-1),\\ X_2 &= (1-\lambda)vy + \lambda(v+y-1),\\ X_3 &= (1-\mu)uv + \mu(u+v-1),\\ X_4 &= (1-\mu)xy + \mu(x+y-1), \end{aligned}$$

so that

$$\begin{aligned} A &= (1-\lambda)\max(0,X_3)\max(0,X_4) + \lambda(\max(0,X_3)+\max(0,X_4)-1),\\ B &= (1-\mu)\max(0,X_1)\max(0,X_2) + \mu(\max(0,X_1)+\max(0,X_2)-1), \end{aligned}$$

and then arrive at the following equivalent formulation of our problem

$$\begin{aligned} &\forall\, x,y,u,v \in \mathbb{R} : \Big((0<\lambda<\mu<\infty) \ \wedge\ (0<x<1) \ \wedge\ (0<y<1) \ \wedge\ (0<u<1) \ \wedge\ (0<v<1)\Big)\\ &\Rightarrow \Big(\Big(X_1 \le 0 \ \wedge\ X_2 \le 0 \ \wedge\ (1-\mu)\cdot 0\cdot 0 + \mu(0+0-1) \le 0\\ &\quad \vee\ X_1 > 0 \ \wedge\ X_2 \le 0 \ \wedge\ (1-\mu)X_1\cdot 0 + \mu(X_1+0-1) \le 0\\ &\quad \vee\ X_1 \le 0 \ \wedge\ X_2 > 0 \ \wedge\ (1-\mu)\cdot 0\cdot X_2 + \mu(0+X_2-1) \le 0\\ &\quad \vee\ X_1 > 0 \ \wedge\ X_2 > 0 \ \wedge\ (1-\mu)X_1X_2 + \mu(X_1+X_2-1) \le 0\Big) \quad \Big\}_{B\le 0}\\ &\ \vee\ \Big(X_1 \le 0 \ \wedge\ X_2 \le 0 \ \wedge\ X_3 \le 0 \ \wedge\ X_4 \le 0\\ &\qquad \wedge\ (1-\lambda)\cdot 0\cdot 0 + \lambda(0+0-1) \ge (1-\mu)\cdot 0\cdot 0 + \mu(0+0-1) > 0\\ &\quad \vee\ X_1 > 0 \ \wedge\ X_2 \le 0 \ \wedge\ X_3 \le 0 \ \wedge\ X_4 \le 0\\ &\qquad \wedge\ (1-\lambda)\cdot 0\cdot 0 + \lambda(0+0-1) \ge (1-\mu)X_1\cdot 0 + \mu(X_1+0-1) > 0\\ &\quad \vee\ \cdots\\ &\quad \vee\ X_1 > 0 \ \wedge\ X_2 > 0 \ \wedge\ X_3 > 0 \ \wedge\ X_4 > 0\\ &\qquad \wedge\ (1-\lambda)X_3X_4 + \lambda(X_3+X_4-1) \ge (1-\mu)X_1X_2 + \mu(X_1+X_2-1) > 0\Big)\Big) \quad \Big\}_{A\ge B>0}. \end{aligned}$$

The subformula corresponding to $A \ge B > 0$ consists altogether of 16 clauses of which only three are shown here to indicate the pattern.

3. *Discard redundant clauses.* The number of clauses in this last formula can be reduced considerably. For example, from

$$(0<\lambda<\mu<\infty) \ \wedge\ (0<x<1) \ \wedge\ (0<y<1) \ \wedge\ (0<u<1) \ \wedge\ (0<v<1) \ \wedge\ (X_1 \le 0) \ \wedge\ (X_2 \le 0)$$

it follows that $(1-\mu)\cdot 0\cdot 0 + \mu(0+0-1) = -\mu \le 0$ is trivially true, so that the first clause simplifies to $X_1 \le 0 \ \wedge\ X_2 \le 0$.
Similarly, the second and the third clause simplify to $X_1 > 0 \ \wedge\ X_2 \le 0$ and $X_1 \le 0 \ \wedge\ X_2 > 0$, respectively. (CAD computations confirm these assertions quickly.) The last literals of the first three clauses dropped, we can simplify the first four clauses, corresponding to $B \le 0$, to

$$\begin{aligned} &(X_1 \le 0 \ \wedge\ X_2 \le 0) \ \vee\ (X_1 > 0 \ \wedge\ X_2 \le 0) \ \vee\ (X_1 \le 0 \ \wedge\ X_2 > 0)\\ &\qquad \vee\ (X_1 > 0 \ \wedge\ X_2 > 0 \ \wedge\ (1-\mu)X_1X_2 + \mu(X_1+X_2-1) \le 0)\\ &\iff \neg(X_1 > 0 \ \wedge\ X_2 > 0) \ \vee\ (X_1 > 0 \ \wedge\ X_2 > 0 \ \wedge\ (1-\mu)X_1X_2 + \mu(X_1+X_2-1) \le 0)\\ &\iff X_1 \le 0 \ \vee\ X_2 \le 0 \ \vee\ (1-\mu)X_1X_2 + \mu(X_1+X_2-1) \le 0. \end{aligned}$$

The simplification of the remaining 16 clauses is complementary: here, all but the last simplify to false, and thus these clauses can be dropped altogether. For verifying that a clause

$$\begin{aligned} &(0<\lambda<\mu<\infty) \ \wedge\ (0<x<1) \ \wedge\ (0<y<1) \ \wedge\ (0<u<1) \ \wedge\ (0<v<1)\\ &\quad \wedge\ (X_1 \diamond 0) \ \wedge\ (X_2 \diamond 0) \ \wedge\ (X_3 \diamond 0) \ \wedge\ (X_4 \diamond 0) \ \wedge\ (A \ge B > 0), \end{aligned}$$

is unsatisfiable, with $\diamond$ denoting either $\leq$ or $>$, it is sufficient to show unsatisfiability of the clause with $A \geq B > 0$ replaced by the weaker conditions $A \geq 0$ or $B \geq 0$. For 15 of the 16 clauses, a CAD computation quickly yields false for at least one of these two choices. The only surviving clause is the one corresponding to $X_1 > 0 \ \wedge \ X_2 > 0 \ \wedge \ X_3 > 0 \ \wedge \ X_4 > 0$. Dropping all the others and taking into account also the simplified form of the first four clauses, we arrive at the equivalent formulation

$$\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : \Big((0 < \lambda < \mu < \infty) \ \wedge \ (0 < x < 1) \ \wedge \ (0 < y < 1) \ \wedge \ (0 < u < 1) \ \wedge \ (0 < v < 1)\Big) \\
&\Rightarrow \Big(X_1 \leq 0 \ \vee \ X_2 \leq 0 \ \vee \ (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0 \\
&\qquad \vee \ X_1 > 0 \ \wedge \ X_2 > 0 \ \wedge \ X_3 > 0 \ \wedge \ X_4 > 0 \\
&\qquad\quad \wedge \ (1-\lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \geq (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0\Big).
\end{aligned}$$

4. *Apply some logical simplifications.* First of all, we may drop the conditions $X_1 > 0$ and $X_2 > 0$ from the last clause because $X_1 \leq 0$ and $X_2 \leq 0$ are part of the disjunction. Furthermore, because of

$$\begin{aligned}
&\Big((0 < \lambda < \mu < \infty) \ \wedge \ (0 < x < 1) \ \wedge \ (0 < y < 1) \ \wedge \ (0 < u < 1) \ \wedge \ (0 < v < 1) \\
&\quad \wedge \ (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0\Big) \Rightarrow X_i > 0 \qquad (i = 1, 2, 3, 4)
\end{aligned}$$

as confirmed by a CAD computation within a few minutes, also the parts $X_3 > 0$ and $X_4 > 0$ in the last clause are redundant and can be dropped. Moreover, $(1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0 \Rightarrow X_i > 0$ $(i = 1, 2)$ is equivalent to $X_i \leq 0 \Rightarrow (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0$ $(i = 1, 2)$ which allows us to discard $X_1 \leq 0$ and $X_2 \leq 0$ from the disjunction.
Dropping also the > 0 at the very end of the last clause, which is allowed because $(1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0$ appears in the disjunction, we arrive at the equivalent formulation

$$\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : \Big((0 < \lambda < \mu < \infty) \ \wedge \ (0 < x < 1) \ \wedge \ (0 < y < 1) \ \wedge \ (0 < u < 1) \ \wedge \ (0 < v < 1)\Big) \\
&\quad \Rightarrow \Big((1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0 \\
&\qquad \vee \ (1-\lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \geq (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1)\Big).
\end{aligned}$$

5. *Apply some algebraic simplifications.* In terms of x, y, u, v we have for the second inequality

$$\begin{aligned}
&((1-\lambda)X_3X_4 + \lambda(X_3 + X_4 - 1)) - ((1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1)) \\
&= (\mu - \lambda)((\mu + \lambda(1-\mu))(u-1)(v-1)(x-1)(y-1) \\
&\qquad - ((u-1)y - u)((v-1)x - v) + 1) \geq 0,
\end{aligned}$$

from which the factor $(\mu - \lambda)$ can be discarded because $0 < \lambda < \mu < \infty$ is part of the assumptions. Replacing x, y, u, v by $1-x, 1-y, 1-u, 1-v$, respectively, turns the last inequality into

$$uy + vx(1 - (1-\mu)(1-\lambda)uy) \geq 0.$$

The substitutions leave the conditions $0 < x < 1$, $0 < y < 1$, $0 < u < 1$, $0 < v < 1$ invariant but turn the first inequality $(1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0$ into

$$\begin{aligned}
&u((\lambda-1)x + 1)((\mu-1)((\lambda-1)vy + v + y) + 1) \\
&\quad + (\mu-1)x((\lambda-1)vy + v + y) + ((\lambda-1)vy + v + y) + x - 1 \geq 0.
\end{aligned}$$

This can be simplified further by replacing the subexpression $(\lambda-1)vy + v + y$ by a new variable $\tilde{v}$, viz. by additionally substituting v by $(\tilde{v} - y)/(1 + (\lambda-1)y)$. This turns the first inequality into

$$u((\lambda-1)x + 1)((\mu-1)\tilde{v} + 1) + (\mu-1)\tilde{v}x + \tilde{v} + x - 1 \geq 0$$

and the second into

$$\frac{\tilde{v}x(1-(\lambda-1)(\mu-1)uy)+y((\lambda-1)uy((\mu-1)x+1)+u-x)}{(\lambda-1)y+1}\geq 0$$

The denominator can be cleared because $(\lambda-1)y+1>0$ is a consequence of the assumptions. The substitution also turns the condition $0<v<1$ into $y<\tilde{v}<1+\lambda y$. Putting things together, we arrive at the equivalent formulation

$$\begin{aligned}\forall\, x,y,u,\tilde{v}:\ &\Big((0<\lambda<\mu<\infty)\ \wedge\ (0<x<1)\ \wedge\ (0<y<1)\ \wedge\ (0<u<1)\ \wedge\ (y<\tilde{v}<1+\lambda y)\Big)\\ &\Rightarrow (u((\lambda-1)x+1)((\mu-1)\tilde{v}+1)+(\mu-1)\tilde{v}x+\tilde{v}+x-1\geq 0\\ &\quad\vee\ \ \tilde{v}x(1-(\lambda-1)(\mu-1)uy)+y((\lambda-1)uy((\mu-1)x+1)+u-x)\geq 0).\end{aligned}$$

With this last formulation, the quantifier elimination problem can be completed automatically within a reasonable amount of time, at least if it is properly input. The order of the quantifiers, while logically irrelevant, has a dramatic influence on the runtime. We found that a feasible order is $\mu,\lambda,u,y,\tilde{v},x$. Mathematica's command Resolve unfortunately reorders the quantifiers internally, in this case not to the advantage of the performance. So we have to do the elimination by resorting to the low-level CAD command. It is also advantageous to consider the negation of the whole formula, and then taking the complement of the result to obtain the desired region for μ,λ. We thus consider the quantified formula

$$\begin{aligned}\exists\, x,y,u,\tilde{v}:\ &\Big((0<\lambda<\mu<\infty)\ \wedge\ (0<x<1)\ \wedge\ (0<y<1)\ \wedge\ (0<u<1)\ \wedge\ (y<\tilde{v}<1+\lambda y)\Big)\\ &\wedge\ u((\lambda-1)x+1)((\mu-1)\tilde{v}+1)+(\mu-1)\tilde{v}x+\tilde{v}+x-1<0\\ &\wedge\ \tilde{v}x(1-(\lambda-1)(\mu-1)uy)+y((\lambda-1)uy((\mu-1)x+1)+u-x)<0.\end{aligned}$$

To further improve the performance, we consider the cases $0<\lambda\leq 1$ and $\lambda>1$ separately. For $0<\lambda\leq 1$, the body of the existentially quantified formula is unsatisfiable, even when the first big inequality is dropped: A CAD computation quickly asserts that

$$\begin{aligned}&\Big((0<\lambda<1)\ \wedge\ (\lambda<\mu<\infty)\ \wedge\ (0<x<1)\ \wedge\ (0<y<1)\ \wedge\ (0<u<1)\ \wedge\ (y<\tilde{v}<1+\lambda y)\Big)\\ &\quad\wedge\ \tilde{v}x(1-(\lambda-1)(\mu-1)uy)+y((\lambda-1)uy((\mu-1)x+1)+u-x)<0\end{aligned}$$

is equivalent to false. For $\lambda>1$, we proceed in two steps. First we compute a CAD only for

$$\begin{aligned}&\Big((1<\lambda<\mu<\infty)\ \wedge\ (0<x<1)\ \wedge\ (0<y<1)\ \wedge\ (0<u<1)\ \wedge\ (y<\tilde{v}<1+\lambda y)\Big)\\ &\quad\wedge\ u((\lambda-1)x+1)((\mu-1)\tilde{v}+1)+(\mu-1)\tilde{v}x+\tilde{v}+x-1<0.\end{aligned}$$

This takes about a minute and then gives something which is trivially equivalent to

$$(1<\lambda<\mu<\infty)\ \wedge\ (0<u<1)\ \wedge\ \left(0<y<\tilde{v}<\frac{1-u}{1+(\mu-1)u}\right)\ \wedge\ \left(0<x<\frac{1-u-\tilde{v}-(\mu-1)u\tilde{v}}{(1+(\lambda-1)u)(1+(\mu-1)\tilde{v})}\right).$$

Denoting this latter formula by Φ, we then compute the CAD of

$$\Phi\ \wedge\ \tilde{v}x(1-(\lambda-1)(\mu-1)uy)+y((\lambda-1)uy((\mu-1)x+1)+u-x)<0.$$

This takes about three minutes and then returns

$$\left(\mu>17+12\sqrt{2}\right)\ \wedge\ \left(\left(\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}}\right)^2<\lambda<\mu<\infty\right)\ \wedge\ (\ldots)$$

where $(\ldots)$ is some messy formula involving $u,y,\tilde{v},x$. The specification of the CAD algorithm implies now that the existentially quantified formula above is valid if and only if μ and λ satisfy this formula with the $(\ldots)$ part removed.

Intersecting the complement of this region with the region where $0 < \lambda < \mu < \infty$ (another quick CAD computation), we finally obtain

$$\left(0 < \mu \leq 17 + 12\sqrt{2} \ \wedge \ 0 < \lambda < \mu < \infty\right) \ \vee \ \left(\mu > 17 + 12\sqrt{2} \ \wedge \ 0 < \lambda \leq \left(\frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}}\right)^2\right)$$

as claimed in the beginning.

We repeat that although the CAD algorithm does not deliver a "human-readable" proof along with its output, the correctness of the algorithm implies that its output has proof character according to the highest logical standards. In addition, it is important to observe that we did not only *prove* Theorem 2 computationally (in the sense of entering a conjecture and asking the computer, "is this the right region?"), but we even *discovered* it computationally (in the sense of asking the computer, "what is the right region?").

4. Discussion of results

4.1. Equivalent result

It is interesting to see that Theorem 2 can be expressed in the following equivalent way. This shows that the partial results obtained by the sufficient conditions related to the generalized Mulholland inequality as displayed in Proposition 1 already covered the first four conditions. Condition *(v)* now closes the missing gap for a full characterization of dominance between two Sugeno-Weber t-norms.

Theorem 4. *Consider the family of Sugeno-Weber t-norms* $(T_\lambda^{\mathbf{SW}})_{\lambda\in[0,\infty]}$. *Then, for all* $\lambda, \mu \in [0, \infty]$, $T_\lambda^{\mathbf{SW}}$ *dominates* $T_\mu^{\mathbf{SW}}$ *if and only if one of the following conditions holds:*

(i) $\lambda = 0$,

(ii) $\mu = \infty$,

(iii) $\lambda = \mu$,

(iv) $0 < \lambda < \min(\mu, 1)$,

(v) $0 < \lambda < \mu$ *and* $1 + \sqrt{\lambda\mu} \leq 3(\sqrt{\lambda} + \sqrt{\mu})$.

4.2. Solution sets and their properties

It is further worth to look a bit in more detail at the relationship between the parameters of t-norms T_β being dominated by some T_α for some given α of a parametric family of t-norms $(T_\lambda)_{\lambda\in I}$. Figure 2 visualizes the set $\mathcal{S}$ of all pairs of parameters (λ, μ) such that $T_\lambda^{\mathbf{SW}}$ dominates $T_\mu^{\mathbf{SW}}$, i.e., $\mathcal{S} = \{(\lambda, \mu) \mid T_\lambda \gg T_\mu\}$. We call such a set *solution set*. In a completely analogous way we have illustrated the solution sets for the parametric families of t-norms as summarized in Table 1. The results are displayed in Figure 2 and it is immediately obvious that the solution set of the family of Sugeno-Weber t-norms is much more complex than those of the other families. Note that for the other families we even do have nice Hasse diagrams whereas for the family of Sugeno-Weber t-norms a nice graphic is, at least so far, still missing.

Therefore we inspect the solution set a bit in more detail. The following function is important for the description of the solution set: Let $f\colon\ ]9, \infty[\to \mathbb{R}$ be a function defined, for all $x \in]9, \infty[$, by

$$f(x) = \left(\frac{1 - 3\sqrt{x}}{3 - \sqrt{x}}\right)^2. \tag{2}$$

Then f is involutive, i.e., for all $x \in]9, \infty[$ we have $f(f(x)) = x$. Moreover, since f is continuous and strictly decreasing, its range is a convex set. The boundary limits can be computed as $\lim_{x\to 9} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = 9$ such that $\mathrm{Ran}_f =]9, \infty[$. Moreover, $17 + 12\sqrt{2}$ is the fixpoint of f.

We now study how, for a given α, the set of t-norms $T_\beta^{\mathbf{SW}}$ being dominated by $T_\alpha^{\mathbf{SW}}$ looks like. The Corollaries deal with the different cases for α and such emphasize different aspects in the dominance relationship between the members of the family of Sugeno-Weber t-norms

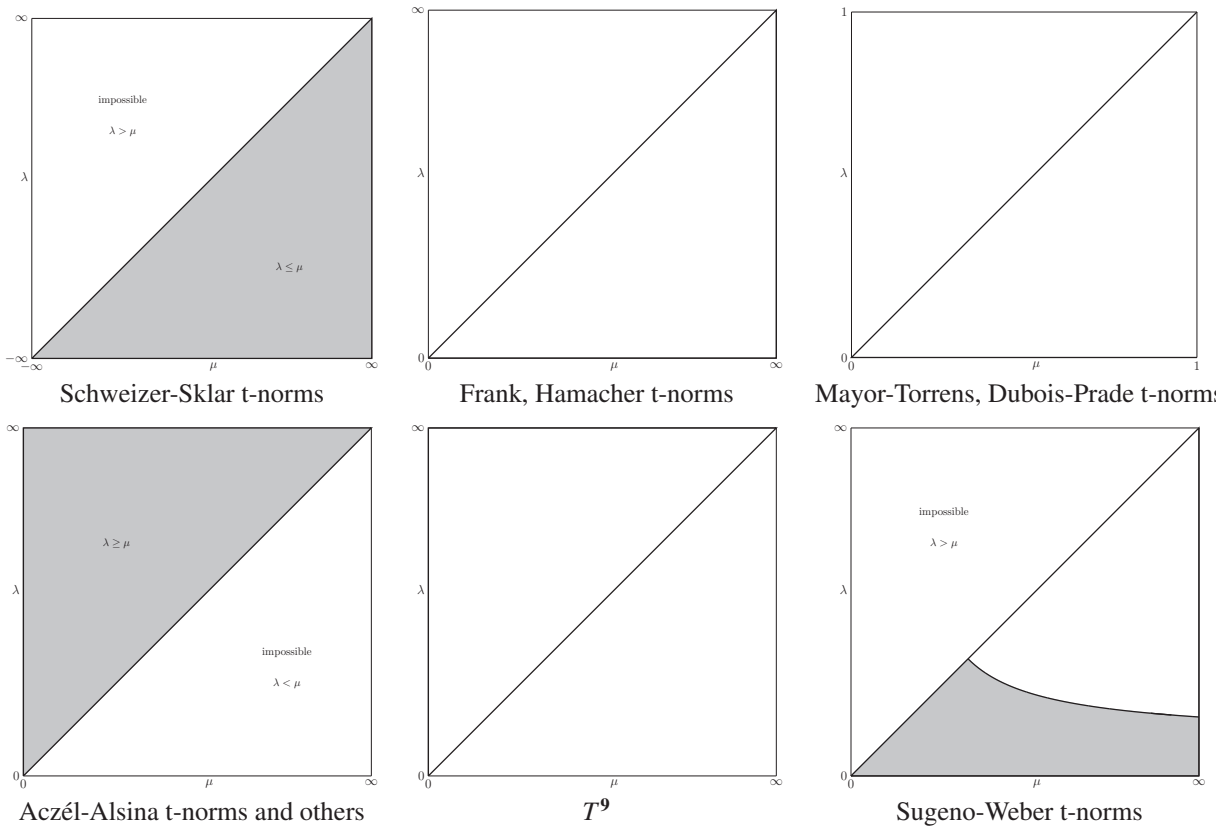

Figure 2: Schematics for the solution sets for the parametric families of t-norms displayed in Tab. 1 and for the family of Sugeno-Weber t-norms.

Corollary 5. *For all α with $0 \leq \alpha \leq 9$ it holds that $T_\alpha^{\mathbf{SW}}$ dominates $T_\beta^{\mathbf{SW}}$ for all $\beta \geq \alpha$.*

Proof. Let α be an arbitrary real number from $[0, 9]$ and choose an arbitrary $\beta \geq \alpha$. If $\alpha = \beta$ or $\beta = \infty$, the dominance relationship trivially holds. We therefore assume that $\alpha < \beta < \infty$. If $\beta \leq 17 + 12\sqrt{2}$ then $T_\alpha^{\mathbf{SW}}$ dominates $T_\beta^{\mathbf{SW}}$ because of Theorem 2 (iv). If $\beta > 17 + 12\sqrt{2}$, then $f(\beta) > 9$ and therefore $f(\beta) > \alpha$ such that $T_\alpha^{\mathbf{SW}}$ dominates $T_\beta^{\mathbf{SW}}$ because of Theorem 2 (v). □

Therefore for all $\alpha \in [0, 9]$ the set $\mathcal{D}_\alpha = \{\beta \mid T_\alpha^{\mathbf{SW}} \gg T_\beta^{\mathbf{SW}}\}$ is of the form $[\alpha, \infty]$. In case that α is greater than 9 but less that $17 + 12\sqrt{2}$ the set $\mathcal{D}_\alpha$ equals $[\alpha, f(\alpha)] \cup \{\infty\}$ as the following Corollary shows.

Corollary 6. *For all α with $9 < \alpha < 17 + 12\sqrt{2}$ it holds that*

(i) $\forall \gamma \in [\alpha, f(\alpha)] : T_\alpha^{\mathbf{SW}} \gg T_\gamma^{\mathbf{SW}}$,

(ii) $\forall \delta > f(\alpha) : T_\alpha^{\mathbf{SW}} \gg T_\delta^{\mathbf{SW}} \iff \delta = \infty$.

Proof. Consider some α such that $9 < \alpha < 17 + 12\sqrt{2}$. Since f is continuous and strictly decreasing it obtains its minimal value at its upper boundary. Since $\alpha \leq 17 + 12\sqrt{2}$ it follows that $f(\alpha) \geq 17 + 12\sqrt{2}$ for all α with $9 < \alpha < 17 + 12\sqrt{2}$.

(i) Consider some $\gamma \in [\alpha, f(\alpha)]$. If $\alpha \leq \gamma \leq 17 + 12\sqrt{2}$, $T_\alpha^{\mathbf{SW}}$ dominates $T_\gamma^{\mathbf{SW}}$ because of Theorem 2 (iii) and (iv). For $17 + 12\sqrt{2} < \gamma \leq f(\alpha)$ the decreasingness and involutivness of f imply that $\alpha = f(f(\alpha)) \leq f(\gamma)$ such that $T_\alpha^{\mathbf{SW}} \gg T_\gamma^{\mathbf{SW}}$ due to Theorem 2 (v).

(ii) Consider some $\delta > f(\alpha)$ then if $\delta = \infty$, $T_\alpha^{\mathbf{SW}} \gg T_\delta^{\mathbf{SW}}$ trivially holds. Vice versa if $T_\alpha^{\mathbf{SW}}$ dominates $T_\delta^{\mathbf{SW}}$, then necessarily $\delta = \infty$, since $f(\delta) < \alpha$.

□

Finally for all $\alpha \geq 17 + 12\sqrt{2}$ the set $\mathcal{D}_\alpha$ just consists of α and ∞.

Corollary 7. *For all* $\alpha, \beta \geq 17 + 12\sqrt{2}$*:*

$$T_\alpha^{\mathbf{SW}} \gg T_\beta^{\mathbf{SW}} \quad \Rightarrow \quad \alpha = \beta \vee \max(\alpha, \beta) = \infty.$$

These results allow for an alternative proof of the transitivity of dominance in the family of Sugeno-Weber t-norms.

4.3. Alternative proof for transitivity

Consider three members $T_a^{\mathbf{SW}}$, $T_b^{\mathbf{SW}}$, $T_c^{\mathbf{SW}}$ of the family of Sugeno-Weber t-norms with arbitrary $a, b, c \in [0, \infty]$. We assume without loss of generality that $a \neq b \neq c \neq a$. Assume that $T_a^{\mathbf{SW}} \gg T_b^{\mathbf{SW}}$ and $T_b^{\mathbf{SW}} \gg T_c^{\mathbf{SW}}$ then $a < b < c$ due to the ordering. We additionally assume that $c < \infty$ for which $T_a^{\mathbf{SW}} \gg T_c^{\mathbf{SW}}$ trivially holds. For showing that indeed also $T_a^{\mathbf{SW}}$ dominates $T_c^{\mathbf{SW}}$ we distinguish the following cases:

Case 1. If $b \geq 17 + 12\sqrt{2}$, then $c \geq 17 + 12\sqrt{2}$ and therefore, because of Corollary 7, $b = c$ or $c = \infty$, the latter being a contradiction.

Case 2. If $9 < b < 17 + 12\sqrt{2}$, then necessarily, and because of Corollary 6, $c \in [b, f(b)[$. Since $a \leq b$, it follows that, for $a > 9$, $f(a) \geq f(b)$ and therefore $c \in [b, f(b)[\subseteq [a, f(a)[$ such that $T_a^{\mathbf{SW}} \gg T_c^{\mathbf{SW}}$ due to Corollary 6. In case $a \leq 9$, $T_a^{\mathbf{SW}}$ dominates $T_c^{\mathbf{SW}}$ trivially due to Corollary 5.

Case 3. If $b \leq 9$, then also $a \leq 9$ such that $T_a^{\mathbf{SW}}$ dominates $T_c^{\mathbf{SW}}$.

In all cases $T_a^{\mathbf{SW}}$ dominates $T_c^{\mathbf{SW}}$ such that the transitivity of dominance in this family is proven.

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