When can we detect that a P-finite sequence is positive?

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ABSTRACT

We consider two algorithms which can be used for proving positivity of sequences that are defined by a linear recurrence equation with polynomial coefficients (P-finite sequences). Both algorithms have in common that while they do succeed on a great many examples, there is no guarantee for them to terminate, and they do in fact not terminate for every input. For some restricted classes of P-finite recurrence equations of order up to three we provide a priori criteria that assert the termination of the algorithms.

Categories and Subject Descriptors

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General Terms

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1. INTRODUCTION

Inequalities for special functions are a serious challenge, both from the traditional paper-and-pencil point of view, but also (and in particular) for computer algebra. In contrast to the vast number of algorithms for dealing with identities, almost no algorithms are available for inequalities. Already for the very restricted class of sequences satisfying linear recurrence equations with constant coefficients (C-finite sequences), the positivity problem leads to hard number theoretic questions to which no solutions are known today, see [8, 10] and the references given there for the current state of the struggle.

Still, inequalities are not entirely hopeless. For example, Mezzarobba and Salvy have recently given an algorithm for effectively computing tight upper bounds for sequences defined by linear recurrence equations with polynomial coefficients (P-finite sequences) [16]. Five years ago, Gerhold and Kauers [11] proposed a method applicable to inequalities concerning quantities that satisfy recurrence equations of a very general type. Their method consists of constructing a sequence of polynomial sufficient conditions that would imply the non-polynomial inequality under consideration. If one of the conditions in the sequence happens to be true (which can be detected, e.g., with Cylindrical Algebraic Decomposition [5, 6, 4, 2]), the method succeeds, otherwise it keeps on running forever. Simultaneously, the method searches for counterexamples and it will find one and terminate for every false inequality.

Despite its simplicity, the method has proven quite successful in applications. Not only did it provide the first computer proofs of some special function inequalities from the literature [11, 12, 13, 14], but it even helped to resolve some open conjectures [1, 15, 14, 17]. At the same time, the method remains somewhat unsatisfactory from a computational point of view, as it is not clear on which inequalities it succeeds and on which it doesn't. It would be interesting to have, at least for some restricted classes, some a priori criteria telling us whether the method (or some variation of it) will succeed or not.

Our goal in this paper is to provide such criteria for two particular proving procedures (Algorithms 1 and 2 described below). We are far from being able to give a full answer to the question posed in the title, but we can identify some nontrivial portions of P-finite recurrence equations of fixed order on which termination of Algorithms 1 or 2 is guaranteed. For first order equations, deciding positivity is trivial. For second order equations, we provide a result (Theorems 2 and 3) that answers the question under a genericity assumption. For third order equations, we are able to identify the terminating cases of Algorithm 2 but only have partial results for Algorithm 1 supplemented by empirical evidence supporting a conjecture concerning its terminating cases. An interesting aspect of our analysis is that algorithms for real quantifier elimination are not only used as a subroutine of Algorithms 1 and 2, but they are also contributing in an essential way to the proofs of our termination theorems. It

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is therefore possible—in principle—to extend our results to equations of order greater than three. Only the increasing time and memory requirements of the computations have prevented us from doing so.

2. PRELIMINARIES

A sequence $f\colon \mathbb{N} \to K := \mathbb{R} \cap \bar{\mathbb{Q}}$ is called *P-finite* (or *holonomic*) if there exist polynomials $p_0, \dots, p_r \in K[x]$, not all zero, such that

$$p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r) = 0$$

for all $n \in \mathbb{N}$. Such an equation is called a (P-finite) *recurrence,* and r is called its order. If $p_r(n) \neq 0$ for all $n \in \mathbb{N}$, then the infinite sequence f is uniquely determined by the recurrence and r *initial values* $f(0), f(1), \dots, f(r-1)$. The assumption $p_r(n) \neq 0$ for all $n \in \mathbb{N}$ can be adopted without loss of generality, because we can substitute $g(n) = f(n+u)$ for some u larger than the greatest integer root of p_r and then consider g instead of f and check nonnegativity of the finitely many terms $f(0), f(1), \dots, f(u-1)$ by inspection. We will do so.

From now on, all recurrences are assumed to have a leading coefficient p_r with no positive integer roots.

A P-finite recurrence is called *balanced* if $\deg p_0 = \deg p_r$ and $\deg p_i \leq \deg p_0$ $(i = 1, \dots, r)$. The *characteristic polynomial* of a balanced recurrence is defined as

$$\operatorname{lc}_y\bigl(p_0(y) + p_1(y)x + p_2(y)x^2 + \cdots + p_r(y)x^r\bigr) \in K[x].$$

Its roots $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ are called the *eigenvalues* of the recurrence. (The λ_i are not necessarily distinct.) Note that for a balanced recurrence, the characteristic polynomial has always degree r and it has never 0 as a root.

An eigenvalue λ_i is called *dominant* if $|\lambda_j| \leq |\lambda_i|$ for all $j = 1, \dots, r$. Dominant eigenvalues govern the asymptotics of the sequences defined by the recurrence [20, 9]. If there is a unique dominant eigenvalue λ_i, then for we will usually have

$$f(n) \sim c(n)\lambda_i^n \quad (n \to \infty)$$

where c is of subexponential growth in the sense that

$$\frac{c(n+1)}{c(n)} \xrightarrow{n\to\infty} 1.$$

There may be choices of initial values for which $c(n) = 0$ for all n so that the asymptotics of f is not affected by λ_i but by the next smaller eigenvalue(s). Whether this is the case or not can be hard to verify formally, but is usually easy to verify empirically. Some of our termination results apply only to this *generic* situation where initial values are chosen such as to actually exhibit the asymptotic behavior predicted by the dominant eigenvalue.

Finally, if the dominant eigenvalue λ_i is not real and positive, then it is clear that f will be ultimately oscillating, and so $f(n) \geq 0$ cannot possibly be true for all n. This case can be sorted out trivially beforehand, and we may therefore assume that the unique dominant eigenvalue is real and positive. In this case, the substitution $g(n) = f(n)/\lambda_i^n$ turns the recurrence

$$p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r) = 0$$

into

$$p_0(n)g(n) + p_1(n)\lambda_i g(n+1) + \cdots + p_r(n)\lambda_i^r g(n+r) = 0$$

whose dominant eigenvalue is 1. As $g(n) \geq 0 \iff f(n) \geq 0$, it suffices to consider this case.

3. INDUCTION BASED PROVING PROCEDURES

3.1 The Original Version

The approach of [11] is as follows. Suppose that $f\colon \mathbb{N} \to K$ is defined by a recurrence

$$p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r) = 0$$

and initial values $f(0) = f_0, f(1) = f_1, \dots, f(r-1) = f_{r-1}$. We seek to prove $f(n) \geq 0$ for all $n \in \mathbb{N}$ by induction:

$$f(n) \geq 0 \land \cdots \land f(n+r-1) \geq 0 \Longrightarrow f(n+r) \geq 0.$$

Because of the recurrence, this is equivalent to

$$\begin{aligned} &f(n) \geq 0 \land \cdots \land f(n+r-1) \geq 0 \\ &\quad\Longrightarrow -\frac{p_0(n)}{p_r(n)}f(n) - \cdots - \frac{p_{r-1}(n)}{p_r(n)}f(n+r-1) \geq 0 \end{aligned}$$

For this to be true for all $n \in \mathbb{N}$, it is sufficient that the *induction step formula*

$$\begin{aligned} &\forall\, y_0, y_1, \dots, y_{r-1} \in \mathbb{R}\ \forall\, x \in \mathbb{R}: \\ &\quad \bigl(x \geq 0 \land y_0 \geq 0 \land \cdots \land y_{r-1} \geq 0\bigr) \\ &\qquad \Longrightarrow -\frac{p_0(x)}{p_r(x)}y_0 - \cdots - \frac{p_{r-1}(x)}{p_r(x)}y_{r-1} \geq 0 \end{aligned}$$

is true, and this can be decided by a quantifier elimination algorithm. If it is true, the induction step is established and f is nonnegative everywhere if and only if it is nonnegative for $n = 0, \dots, r-1$, which can be checked.

In the unlucky case when the induction step formula is false, there is no immediate conclusion about f that could be drawn. In this case, *refined induction step formulas*

$$f(n) \geq 0 \land \cdots \land f(n+\varrho-1) \geq 0 \Longrightarrow f(n+\varrho) \geq 0$$

for $\varrho > r$ are constructed. Using the recurrence, each term $f(n+i)$ can be rewritten as a linear combination of $f(n)$, $\dots$, $f(n+r-1)$ with rational function coefficients, and using this rewriting, the refined induction step formula takes the form

$$\begin{aligned} \Phi(\varrho) := \forall\, &y_0, y_1, \dots, y_{r-1} \in \mathbb{R}\ \forall\, x \in \mathbb{R}: \\ &\bigl(x \geq 0 \land y_0 \geq 0 \land \cdots \land y_{r-1} \geq 0 \\ &\quad \land q_{r,0}(x)y_0 + \cdots + q_{r,r-1}(x)y_{r-1} \geq 0 \\ &\quad \land q_{r+1,0}(x)y_0 + \cdots + q_{r+1,r-1}(x)y_{r-1} \geq 0 \\ &\quad\ \vdots \\ &\quad \land q_{\varrho-1,0}(x)y_0 + \cdots + q_{\varrho-1,r-1}(x)y_{r-1} \geq 0\bigr) \\ &\qquad \Longrightarrow q_{\varrho,0}(x)y_0 + \cdots + q_{\varrho,r-1}(x)y_{r-1} \geq 0, \end{aligned}$$

where the $q_{i,j}$ are some rational functions.

The full method then reads as follows.

Algorithm 1.
Input: A P-finite recurrence of order r and a vector of initial

values defining a sequence $f\colon \mathbb{N} \to K$.
Output: True if $f(n) \geq 0$ for all $n \in \mathbb{N}$, False if $f(n) < 0$ for some $n \in \mathbb{N}$, possibly no output at all.

1 **for** $n = 0$ **to** $r-1$ **do**
2   **if** $f(n) < 0$ **then** **return** False
3 **for** $n = r, r+1, r+2, r+3, \ldots$ **do**
4   **if** $\Phi(n)$ **then** **return** True
5   **if** $f(n) < 0$ **then** **return** False

Example 1. *Let $f\colon \mathbb{N} \to K$ be defined by*

$$\begin{aligned}&(2n+13)f(n+3) - (5n+22)f(n+2)\\ &\quad + (3n+20)f(n+1) - (2n+7)f(n) = 0,\\ &f(0) = f(1) = f(2) = 1.\end{aligned}$$

We use Algorithm 1 to show that $f(n) \geq 0$ for all $n \in \mathbb{N}$.

Since $f(0), f(1), f(2) \geq 0$, we enter the loop in line 3. For $n = 3$, we have

$$\begin{aligned}\Phi(n) = \forall\ &y_0, y_1, y_2\ \forall\ x \in \mathbb{R} :\\ &\bigl(x \geq 0 \land y_0 \geq 0 \land y_1 \geq 0 \land y_2 \geq 0\bigr)\\ &\quad \Longrightarrow \tfrac{2x+7}{2x+13}y_0 - \tfrac{3x+20}{2x+13}y_1 + \tfrac{5x+22}{2x+13}y_2 \geq 0.\end{aligned}$$

This is false, but since $f(3) = 9/13 > 0$ (checked in line 5), we continue.

The formula $\Phi(4)$ is too lengthy to be reproduced here explicitly, and it is also false. Yet $f(4) = 61/195 \geq 0$, so we proceed to consider the even lengthier formula $\Phi(5)$, which turns out to be true. At this point the algorithm terminates with output True.

3.2 A Variation

In cases where Algorithm 1 does not terminate, it is sometimes possible to prove inductively the stronger statement that $f(n)$ is increasing, viz. that $f(n+1) \geq f(n)$ for all $n \geq 0$. While this is obviously a sufficient condition for $f(n) \geq 0$ for all n, there are of course sequences f which are non-negative but not increasing. For such cases, a good strategy is to prove that $\mu^{-n}f(n)$ is increasing, for some suitably chosen constant $\mu > 0$. The choice of μ is critical in two respects: it must be small enough to assure that $\mu^{-n}f(n)$ actually is increasing, and it must be big enough to allow for an inductive proof. The following algorithm proves positivity of a P-finite sequence f by searching for a μ that meets both criteria.

Algorithm 2.
Input: A P-finite recurrence of order r and a vector of initial values defining a sequence $f\colon \mathbb{N} \to K$.
Output: True if $f(n) \geq 0$ for all $n \in \mathbb{N}$, False if $f(n) < 0$ for some $n \in \mathbb{N}$, possibly no output at all.

1 Determine a quantifier free formula $\Phi(\xi, \mu)$ equivalent to

$$\begin{aligned}\forall\ &y_0, \ldots, y_{r-1}\ \forall\ x \geq \xi :\\ &\bigl(y_0 \geq 0 \land y_1 \geq \mu y_0 \land \cdots \land y_{r-1} \geq \mu y_{r-2}\bigr)\\ &\quad \Longrightarrow -\frac{p_0(x)}{p_r(x)}y_0 - \cdots - \frac{p_{r-1}(x)}{p_r(x)}y_{r-1} \geq \mu y_{r-1}\end{aligned}$$

2 **for** $n = 0, 1, 2, 3, \ldots$ **do**
3   **if** $f(n) < 0$ **then**
4   **return** False
5   **else if** $\exists\ \mu \geq 0 : \Phi(n, \mu) \land f(n+1) \geq \mu f(n)$ $\land \cdots \land f(n+r-1) \geq \mu f(n+r-2)$ **then**
6   **return** True

Theorem 1. *Algorithm 2 is correct.*

Proof. Correctness is obvious whenever the algorithm returns False, because this happens only when an explicit point n with $f(n) < 0$ has been found. Suppose now that the algorithm returns True at the nth iteration of the for loop. Then $f(k) \geq 0$ for $k = 0, \ldots, n$, otherwise the algorithm would have terminated in an earlier iteration with output False. The condition in line 5 inductively implies

$$\exists\ \mu \geq 0\ \forall\ k \geq n : f(k+1) \geq \mu f(k).$$

Since $\mu \geq 0$ and $f(n) \geq 0$, this inductively implies $f(k) \geq 0$ also for all $k > n$. $\square$

Example 2. *Let $f\colon \mathbb{N} \to K$ be defined by*

$$\begin{aligned}&(n+3)f(n+3) - (5n+13)f(n+2)\\ &\quad + (5n+12)f(n+1) - (n+2)f(n) = 0,\\ &f(0) = 1,\ \ f(1) = 1/4,\ \ f(2) = 1/10.\end{aligned}$$

Algorithm 1 does not seem to terminate for this sequence. But Algorithm 2 succeeds.

Step 1 produces the quantifier free formula

$$\xi \geq 0 \land \frac{5-\sqrt{5}}{2} \leq \mu \leq \frac{\sqrt{5\xi^2 + 22\xi + 25} + 5\xi + 13}{2(\xi+3)},$$

which we denote $\Phi(\xi, \mu)$. In the iteration of the for loop, we get:

For $n = 0$, since $f(0) = 1 \geq 0$, we check whether

$$0 \geq 0 \land \frac{5-\sqrt{5}}{2} \leq \mu \leq 3 \land \frac{1}{4} \geq \mu \land \frac{1}{10} \geq \frac{\mu}{4}$$

is satisfiable. As it is not, we proceed.

Also $n = 1$ and $n = 2$, we have $f(n) \geq 0$ but there is no $\mu \geq 0$ with

$$\Phi(n, \mu) \land f(n+1) \geq \mu f(n) \land f(n+2) \geq \mu f(n+1).$$

Then for $n = 3$, since $f(3) \geq 0$, we check whether

$$3 \geq 0 \land \frac{5-\sqrt{5}}{2} \leq \mu \leq \frac{28 + 2\sqrt{34}}{12} \land \frac{17}{80} \geq \frac{\mu}{10} \land \frac{247}{400} \geq \frac{17\mu}{80}$$

is satisfiable. As it is satisfiable (e.g., by $\mu = 2$) the algorithm terminates with output True.

The two strategies employed in Algorithms 1 and 2 in the case where a direct proof of the induction step formula fails (prolonging the induction hypothesis in Algorithm 1 versus multiplying with a positivity preserving exponential in Algorithm 2) are independent of each other. It is possible to merge both strategies into a single strategy that simultaneously prolongs the induction hypothesis and inserts a positivity preserving exponential. An algorithm based on such a combined strategy is easily written down, but turns out to be computationally quite expensive on examples. It would be interesting to carry out the termination analysis given below for the combined algorithm, but the quantifier elimination problems arising in this analysis seem currently too hard to be carried out for the combined algorithm.

4. TERMINATING CASES

Both algorithms given in the previous section may fail to terminate. Our goal now is to identify classes of P-finite recurrence equations for which termination can be guaranteed a priori.

4.1 Order One

This case is rather simple and included here merely for the sake of completeness. If $f\colon \mathbb{N} \to K$ satisfies

$$p_0(n)f(n) + p_1(n)f(n+1) = 0,$$

then $f(n) \geq 0$ for all $n \in \mathbb{N}$ if and only if $f(0) \geq 0$ and $-p_0(n)/p_1(n) \geq 0$ for all $n \in \mathbb{N}$. Since sign changes of $-p_0(n)/p_1(n)$ can occur only at the real roots of p_0 or p_1, the only thing we need to do is to find an upper bound $n_0 \in \mathbb{R}$ for the real roots (this can be done), and then check whether $-p_0(n)/p_1(n) \geq 0$ for $n = 0, 1, 2, \dots, n_0 + 1$.

Example 3. *Consider $f\colon \mathbb{N} \to K$ defined via*

$$(3n - 16)f(n) - (3n - 17)f(n+1) = 0, \qquad f(0) = 1.$$

The roots of p_0, p_1 are $16/3$ and $17/3$, respectively, and they are both less than $n_0 = 6$, for instance. Therefore, since $f(n) \geq 0$ for $n = 0, \dots, 6$, we can conclude that $f(n) \geq 0$ for all $n \in \mathbb{N}$.

4.2 Order Two

We now turn to sequences $f\colon \mathbb{N} \to K$ which are defined by a balanced P-finite recurrence of second order,

$$p_2(n)f(n+2) - p_1(n)f(n+1) - p_0(n)f(n) = 0.$$

We assume (without loss of generality) that 1 is a dominant eigenvalue of this recurrence and let $u \in K$ with $|u| < 1$ be such that

$$(x-1)(x-u) = x^2 - (u+1)x - (-u)$$

is the characteristic polynomial of the recurrence. The question is whether Algorithm 1 and Algorithm 2 succeed in proving that $f(n) \geq 0$ for all n. (If this is actually the case; if it is not, then both algorithms will obviously succeed in finding a counterexample.) We will show that termination of Algorithm 1 depends on the sign of u whereas Algorithm 2 (generically) terminates for all u.

Theorem 2. *If $u \in (-1, 0)$, then Algorithm 1 terminates.*

Proof. Rewrite the recurrence in the form

$$f(n+2) = \frac{p_1(n)}{p_2(n)} f(n+1) + \frac{p_0(n)}{p_2(n)} f(n).$$

Since the characteristic polynomial is

$$(x-1)(x-u) = x^2 - (u+1)x - (-u),$$

we have

$$\frac{p_1(n)}{p_2(n)} \xrightarrow{n\to\infty} u + 1 > 0 \quad \text{and} \quad \frac{p_0(n)}{p_2(n)} \xrightarrow{n\to\infty} -u > 0.$$

Therefore,

$$\exists\, n_0 \in \mathbb{N}\ \forall\, n \geq n_0 : \frac{p_1(n)}{p_2(n)} > 0 \wedge \frac{p_0(n)}{p_2(n)} > 0,$$

where we may safely regard n as ranging not only over the integers but over all reals except the roots of p_2. We show that Algorithm 1 terminates after at most n_0 iterations. The previous formula implies

$$\forall\, y_0, y_1\ \forall\, n \geq n_0 : \big(y_0 \geq 0 \wedge y_1 \geq 0\big) \implies \frac{p_0(n)}{p_2(n)} y_0 + \frac{p_1(n)}{p_2(n)} y_1 \geq 0.$$

Substituting $n \mapsto n - n_0$ leads to

$$\forall\, y_0, y_1\ \forall\, n \geq 0 : \big(y_0 \geq 0 \wedge y_1 \geq 0\big) \implies \frac{p_0(n+n_0)}{p_2(n+n_0)} y_0 + \frac{p_1(n+n_0)}{p_2(n+n_0)} y_1 \geq 0.$$

As the variables y_0, y_1 range over all reals, the latter formula will remain true if we apply a substitution

$$y_0 \mapsto r_0(n)y_0 + r_1(n)y_1,$$
$$y_1 \mapsto r_2(n)y_0 + r_3(n)y_1,$$

where $r_0, \dots, r_3$ are some rational functions in n. This gives

$$\begin{aligned}\forall\, y_0, y_1\ \forall\, n \geq 0 : \big(&r_0(n)y_0 + r_1(n)y_1 \geq 0 \\ &\wedge r_2(n)y_0 + r_3(n)y_1 \geq 0\big) \\ \implies &\frac{p_0(n+n_0)r_0(n) + p_1(n+n_0)r_2(n)}{p_2(n+n_0)} y_0 \\ &+ \frac{p_0(n+n_0)r_1(n) + p_1(n+n_0)r_3(n)}{p_2(n+n_0)} y_1 \geq 0.\end{aligned}$$

We are free to further modify this formula, without harming its truth, by imposing arbitrary additional conditions on the left hand side of the implication.

By choosing r_0, r_1, r_2, r_3 such that

$$f(n+n_0) = r_0(n)f(n) + r_1(n)f(n+1)$$
$$f(n+n_0+1) = r_2(n)f(n) + r_3(n)f(n+1)$$

and adding constraints $q_{i,0}(n)y_0 + q_{i,1}(n)y_1 \geq 0$ encoding $f(n+i) \geq 0$ $(i = 0, \dots, n_0 - 1)$ on the hypothesis part, we obtain precisely the formula $\Phi(n_0)$ as defined in Section 3.1 and used in Algorithm 1. As the formula is true, the algorithm terminates in the n_0th iteration (or earlier), as we wanted to show. □

Remark 1. *Algorithm 1 fails to terminate for positive u. To see this, consider the C-finite recurrence*

$$f(n+2) - (u+1)f(n+1) + uf(n) = 0$$

*for some $u \in (0,1)$. If Algorithm 1 applied to this recurrence terminated in the n_0th iteration, for some $n_0 \geq 0$, then the truth of $\Phi(n_0)$ implies that **no** solution $f\colon \mathbb{N} \to K$ of the recurrence can have n_0 consecutive nonnegative terms followed by a negative term. (So that, if n_0 consecutive terms are found nonnegative, all subsequent terms must be nonnegative as well.)*

To see that no such n_0 can exist for the C-finite recurrence above, it is sufficient to construct for every $n_0 \geq 0$ a solution which contains a run of exactly n_0 nonnegative terms. The general solution of the recurrence is $c_0 + c_1 u^n$ for some constants $c_0, c_1 \in K$. It is easily checked that the choice $c_0 = -1$, $c_1 = u^{-n_0+1}$ has the desired property.

The argument extends, at least for generic initial values, to P-finite balanced recurrence equations, using the fact that the recurrence admits two solutions $f_1, f_2\colon \mathbb{N} \to K$ with $f_1(n+1)/f_1(n) \xrightarrow{n\to\infty} 1$ and $f_2(n+1)/f_2(n) \xrightarrow{n\to\infty} u$.

Theorem 3. *If $u \in (-1,1)\setminus\{0\}$, then Algorithm 2 terminates for generic initial values.*

Proof. Consider the set $D_3 \subseteq \mathbb{R}^3$ consisting of all points (c_0, c_1, μ) satisfying

$$0 < \mu < 1 \wedge \mu < c_1 < 2 \wedge \mu(\mu - c_1) < c_0 < 1$$

and the set

$$D_2 := \{\, (c_0, c_1) \in \mathbb{R}^2 : 0 < c_1 < 2 \wedge -\tfrac{1}{4}c_1^2 < c_0 < 1 \,\}.$$

It can be shown by CAD computations that

$$\forall\ (c_0, c_1) \in D_2\ \exists\ \mu \in (0,1) : (c_0, c_1, \mu) \in D_3 \qquad (1)$$

and that

$$\begin{aligned}&\forall\ (c_0, c_1, \mu) \in D_3\ \forall\ y_0, y_1 \in \mathbb{R} : \\ &\quad (y_0 \geq 0 \wedge y_1 \geq \mu y_0) \implies c_0 y_0 + c_1 y_1 \geq \mu y_1. \qquad (2)\end{aligned}$$

Since $\frac{1}{4}(u+1)^2 > u$ for all $u \in (-1,1)$, the set D_2 contains in particular the point $(-u, u+1)$ where u is from the statement of the theorem. Because of (1), there exists $\mu \in (0,1)$ with $(-u, u+1, \mu) \in D_3$. Since D_3 is open, there exists $\varepsilon > 0$ such that

$$U := (-u-\varepsilon, -u+\varepsilon) \times (u+1-\varepsilon, u+1+\varepsilon) \times \{\mu\} \subseteq D_3.$$

Because of

$$\frac{p_0(n)}{p_2(n)} \xrightarrow{n\to\infty} -u \quad \text{and} \quad \frac{p_1(n)}{p_2(n)} \xrightarrow{n\to\infty} u+1,$$

there exists $\xi \in \mathbb{N}$ such that

$$\Bigl(\frac{p_0(n)}{p_2(n)}, \frac{p_1(n)}{p_2(n)}, \mu\Bigr) \in U \subseteq D_3$$

for all $n \geq \xi$. Together with (2), this implies

$$\begin{aligned}&\exists\ \mu \in (0,1)\ \exists\ \xi \in \mathbb{N}\ \forall\ n \geq \xi\ \forall\ y_0, y_1 \in \mathbb{R} : \\ &(y_0 \geq 0 \wedge y_1 \geq \mu y_0) \implies \frac{p_0(n)}{p_2(n)} y_0 + \frac{p_0(n)}{p_2(n)} y_1 \geq \mu y_1.\end{aligned}$$

Therefore, the set

$$C := \{\, (\xi, \mu) \in (0,\infty) \times (0,1) : \Phi(\xi, \mu) \,\}$$

with $\Phi(\xi,\mu)$ as used in Algorithm 2 is not empty.

Fix some point $(\xi, \mu) \in C$. Then it is immediate by the defining formula that also $(\xi', \mu) \in C$ for every $\xi' > \xi$, so

$$(\xi, \infty) \times \{\mu\} \subseteq C.$$

Let now $f\colon \mathbb{N} \to K$ be the sequence to which Algorithm 2 is applied. Then, because the eigenvalues of its defining recurrence are 1 and u and we have $|u| < 1$ and we assume generic initial values, we have

$$\frac{f(n+1)}{f(n)} \xrightarrow{n\to\infty} 1.$$

Since $\mu < 1$, this implies the existence of an index $m \in \mathbb{N}$ such that $f(n) \geq 0$ for all $n \geq m$ and $f(n+1)/f(n) \geq \mu$ for all $n \geq m$, so that we get $f(n+1) \geq \mu f(n)$ for all $n \geq m$.

It follows that the algorithm terminates no later than at iteration $\max(m, \xi)$. $\square$

Remark 2. *The defining inequalities of D_3 in the preceding proof were found by quantifier elimination applied to formula (2) with the first quantifier dropped. This computation as well as the CAD computations referred to in the proof were performed with Mathematica's built-in implementation of CAD [18, 19]. The computation time is negligible for all of them and we are sure that other implementations [7; 3, etc.] would have no problem with them either.*

Example 4. *The restriction to generic initial values in Theorem 3 is essential: let $f\colon \mathbb{N} \to K$ be defined via*

$$\begin{aligned}(n+3)^2 f(n+2) &- \tfrac{1}{2}(n+2)(3n+11) f(n+1) \\ &+ \tfrac{1}{2}(n+4)(n+1) f(n) = 0\end{aligned}$$

and $f(0) = 1$, $f(1) = 1/4$. Then we have $f(n) = 2^{-n}/(n+1)$ for all $n \in \mathbb{N}$ and so in particular $f(n) \geq 0$ $(n \in \mathbb{N})$.

Algorithm 2 finds

$$\Phi(\xi, \mu) \equiv \tfrac{1}{2} \leq \mu \leq 1 \wedge \xi \geq \frac{17 - 12\mu - \sqrt{25 - 16\mu}}{2\mu - 3}$$

in Step 1 and continues by searching for an index n with $f(n+1) \geq \frac{1}{2} f(n)$. As no such index exist, the search continues forever.

The general solution of the defining recurrence for f is

$$c_0 + c_1 \frac{2^{-n}}{n+1}$$

and we will have $c_0 \neq 0$ for a generic choice of initial values. In these cases, the solution converges to c_0 and therefore eventually reaches an index n_0 with a term that is greater than half its predecessor.

4.3 Order Three

Consider now sequences $f\colon \mathbb{N} \to K$ defined by a balanced P-finite recurrence of third order,

$$p_3(n) f(n+3) - p_2(n) f(n+2) - p_1(n) f(n+1) - p_0(n) f(n) = 0.$$

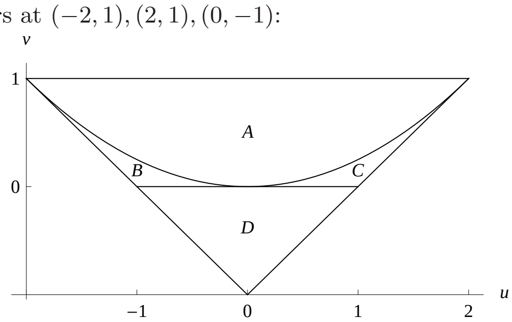
Again, we assume without loss of generality that 1 is a dominant eigenvalue and we let $u, v \in K$ be such that

$$(x-1)(x^2 + ux + v) = x^3 - (1-u)x^2 - (u-v)x - v$$

is the characteristic polynomial of the recurrence under consideration. The condition that the two roots of the quadratic factor belong to the interior of the complex unit disc translates into the condition

$$|u| - 1 < v < 1$$

for the coefficients of the polynomial. The points $(u,v) \in K^2$ satisfying this condition form the interior of the triangle with corners at $(-2,1), (2,1), (0,-1)$:



Just for the sake of orientation: the polynomial $x^2 + ux + v$ has two complex conjugate roots in region A, two positive real roots in region B, two negative real roots in region C, and a positive as well as a negative root in region D.

We want to identify regions of the triangle corresponding to recurrence equations on which Algorithms 1 and 2 terminate. Only for Algorithm 2 we have a satisfactory result, so let us consider this case first.

Theorem 4. *If* $|u|-1 < v < 1$ *and* $4v < (u+1)^2$ *and* $u < 1$*, then Algorithm 2 terminates for generic initial values.*

Proof. Consider the set $D_4 \subseteq \mathbb{R}^4$ consisting of all points (c_0, c_1, c_2, μ) satisfying

$$0 < \mu < 1 \land \mu < c_2 \land \mu(\mu - c_2) < c_1 \land \mu^3 - c_2\mu^2 - c_1\mu < c_0$$

and the set $D_3 \subseteq \mathbb{R}^3$ consisting of all points (c_0, c_1, c_2) with

$$\begin{aligned}&\bigl(0 < c_2 < 2 \land -\tfrac{1}{4}c_2^2 < c_1 < \min(3 - 2c_2, c_2^2)\\ &\quad \land 2c_2^3 + 9c_1c_2 + 27c_0 + 2(c_2^2 + 3c_1)^{3/2} > 0\bigr)\\ &\lor \bigl(0 < c_2 < 1 \land c_1 \geq c_2^2 \land c_0 + c_1c_2 > 0\bigr).\end{aligned}$$

The following facts can be verified by CAD:

- $\forall\ (c_0, c_1, c_2) \in D_3\ \exists\ \mu \in (0,1) : (c_0, c_1, c_2, \mu) \in D_4$
- $\forall\ (c_0, c_1, c_2, \mu) \in D_4\ \forall\ y_0, y_1, y_2 \in \mathbb{R} :$
  $$\begin{aligned}&(y_0 \geq 0 \land y_1 \geq \mu y_0 \land y_2 \geq \mu y_1)\\ &\quad\Longrightarrow c_0y_0 + c_1y_1 + c_2y_2 \geq \mu y_2\end{aligned}$$
- $\forall\ u, v \in \mathbb{R} : \bigl(|u| - 1 < v < 1 \land 4v < (u+1)^2 \land u < 1\bigr)$
  $$\Longrightarrow (v, u - v, 1 - u) \in D_3.$$

Consequently, for u, v from the statement of the theorem there exists $\mu \in (0,1)$ such that $(v, u - v, 1 - u, \mu) \in D_4$. Since D_4 is open, there exists $\varepsilon > 0$ such that

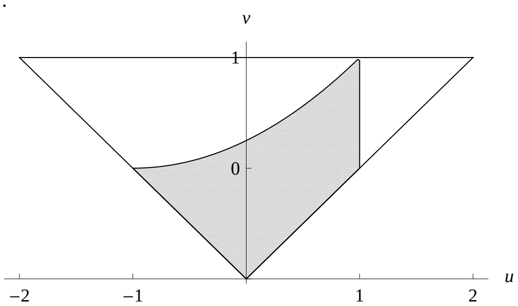
$$\begin{aligned}U :=&\ (v - \varepsilon, v + \varepsilon) \times (u - v - \varepsilon, u - v + \varepsilon)\\ &\times (1 - u - \varepsilon, 1 - u + \varepsilon) \times \{\mu\} \subseteq D_4.\end{aligned}$$

Using

$$\frac{p_0(n)}{p_3(n)} \xrightarrow{n\to\infty} v, \quad \frac{p_1(n)}{p_3(n)} \xrightarrow{n\to\infty} u - v, \quad \frac{p_2(n)}{p_3(n)} \xrightarrow{n\to\infty} 1 - u,$$

the rest of the proof is fully analogous to the proof of Theorem 3. $\square$

The set of points (u, v) for which Theorem 4 asserts termination of Algorithm 2 is the shaded area in the figure below.



The truth of the formula

$$\begin{aligned}&\forall\ u \in (-1,1)\ \forall\ v \in (|u| - 1, 1) :\\ &\quad \bigl[\exists\ \mu > 0\ \forall\ y_0, y_1, y_2 :\\ &\qquad (y_0 \geq 0 \land y_1 \geq \mu y_0 \land y_2 \geq \mu y_1)\\ &\qquad\quad \Longrightarrow v y_0 + (u - v)y_1 + (1 - u)y_2 \geq \mu y_2\bigr]\\ &\quad \Longrightarrow u < 1 \land 4v < (u+1)^2\end{aligned}$$

(as confirmed, once again, by a CAD computation) asserts that Theorem 4 is sharp.

We are not able to provide a sharp result for the terminating region of Algorithm 1. If we proceed to reason as in the proof of Theorem 2, we obtain termination for (u, v) restricted to the (open) triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, essentially because of

$$\begin{aligned}&\forall\ u \in (0,1)\ \forall\ v \in (0,u)\ \forall\ y_0, y_1, y_2 :\\ &\quad (y_0 \geq 0 \land y_1 \geq 0 \land y_2 \geq 0)\\ &\qquad \Longrightarrow v y_0 + (u - v)y_1 + (1 - u)y_2 \geq 0\end{aligned}$$

and the convergences

$$\frac{p_0(n)}{p_3(n)} \xrightarrow{n\to\infty} v, \quad \frac{p_1(n)}{p_3(n)} \xrightarrow{n\to\infty} u - v, \quad \frac{p_2(n)}{p_3(n)} \xrightarrow{n\to\infty} 1 - u.$$

But this is not the entire terminating region. A larger portion of the terminating region can be identified by starting out with a formula corresponding to an induction hypothesis of length four. As the formula

$$\begin{aligned}&\forall\ y_0, y_1, y_2 : (y_0 \geq 0 \land y_1 \geq 0 \land y_2 \geq 0\\ &\qquad \land v y_0 + (u - v)y_1 + (1 - u)y_2 \geq 0)\\ &\quad \Longrightarrow (1 - u)v y_0 + u(1 - u + v)y_1 + (1 - u + u^2 - v)y_2 \geq 0\end{aligned}$$

is true for all (u, v) with

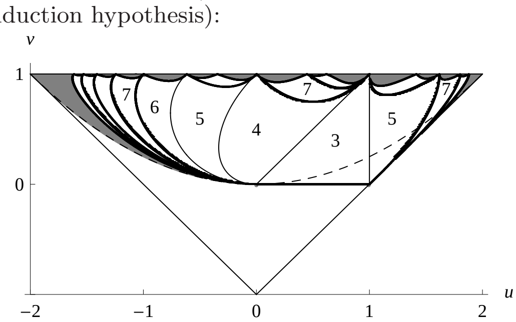
$$\begin{aligned}&u < 1 \land v > 0 \land 1 - u + u^2 - v > 0\\ &\quad \land \bigl(u > 0 \lor u^2 - v - uv + v^2 < 0\bigr),\end{aligned}$$

and as we have

$$\begin{aligned}\frac{p_0(n)p_2(n+1)}{p_3(n)p_3(n+1)} &\xrightarrow{n\to\infty} (1 - u)v,\\ \frac{p_1(n)p_2(n+1) + p_0(n+1)p_3(n)}{p_3(n)p_3(n+1)} &\xrightarrow{n\to\infty} u(1 - u + v),\\ \frac{p_2(n)p_2(n+1) + p_1(n+1)p_3(n)}{p_3(n)p_3(n+1)} &\xrightarrow{n\to\infty} 1 - u + u^2 - v,\end{aligned}$$

Algorithm 1 also terminates for all (u, v) satisfying the conditions stated above.

Starting out with a formula corresponding to a hypothesis of length five leads to a portion of the termination region whose description can be computed in a reasonable amount of time, but which is already too big to be reproduced here. For longer induction hypotheses, the computational effort for doing quantifier elimination becomes prohibitive. But it is still possible to determine experimentally the regions obtained by taking a particular length ϱ of the induction hypothesis taken as the starting point of the termination proof. The empiric results for induction hypotheses of length up to 10 are as follows (the numbers indicate the length of the induction hypothesis):



The picture suggests the following characterization for the full region of termination.

Conjecture 1. *If* $|u| - 1 < v < 1$ *and*

$$(u > 1 \wedge v > 0) \vee 4v > (u+1)^2,$$

then Algorithm 1 terminates.

The conjecture is equivalent to saying that Algorithm 1 terminates if $x^2 + ux + v$ has no positive root. If the conjecture is true, then about 96.35% of the area of the triangle are covered by one of the two algorithms we considered.

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