A MATHEMATICA PACKAGE FOR q-HOLONOMIC SEQUENCES AND POWER SERIES

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Abstract. We describe a Mathematica package for dealing with q-holonomic sequences and power series. The package is intended as a q-analogue of the Maple package gfun and the Mathematica package GeneratingFunctions. It provides commands for addition, multiplication, and substitution of these objects, for converting between various representations (q-differential equations, q-recurrence equations, q-shift equations), for computing sequence terms and power series coefficients, and for guessing recurrence equations given initial terms of a sequence.

1. Introduction

Computer algebra packages for dealing with recurrence equations are of great relevance as they make a lot of time-consuming, tedious and errorprone hand computations obsolete. This is why, for instance, computer algebra implementations of algorithms for hypergeometric summation [16, 15, 1] enjoy a large and still increasing community of users with a broad variety of different backgrounds. Even more so than in the ordinary case, computations in q-calculus tend to involve large expressions that can be tackled by computer algebra much faster and much more reliably than by hand. Algorithms for ordinary hypergeometric summation have therefore been generalized to summation for q-hypergeometric terms [22, 9, 14, 17, 18], and implementations of these algorithms are also widely used.

In contrast, no software package was available so far that would deal with univariate q-holonomic sequences and power series like the Maple package gfun [20] or its Mathematica cousin GeneratingFunctions [11] do for ordinary univariate holonomic sequences and power series. The class of these objects is closed under a certain number of operations, and there are algorithms for computing a recurrence/differential equation for the result of such an operation given recurrences/differential equations of the operands ("executing closure properties"). These algorithms are at the heart of gfun and GeneratingFunctions.

The packages have become an indispensable tool for proving special function identities via computer algebra. They proved particularly useful in connection with hypergeometric summation algorithms, but are also used frequently in experimental mathematics, e.g., for automatically guessing recurrence equations from sample values of otherwise unknown sequences. Naturally, a q-version of gfun/GeneratingFunctions could considerably facilitate the work with q-holonomic objects along the same lines.

It follows from general considerations in Ore rings [5] that the closure properties as well as the corresponding algorithms carry over to the q-case. Direct proofs were also given recently by Koepf et al. [8]. The purpose of the present paper is to present a Mathematica implementation of these and other algorithms for univariate q-holonomic sequences and power series. Besides executing closure properties, our package allows for converting different representations of sequences and power series to each other, for computing sequence values or coefficients of power series, and for automatically guessing q-recurrence equations of q-holonomic sequences given by some finite sample of values. Our package is available for download from

http://www.risc.uni-linz.ac.at/research/combinat/software/

* Partially supported by the Austrian Science Foundation (FWF) grants SFB F1305 and P19462-N18.

† Partially supported by the Austrian Science Foundation (FWF) grants SFB F1305.

In Sections 2–4 of this paper, we give descriptions of the commands provided by our package, along with simple examples and the relevant theoretical background. In Section 5, we present a collection of some typical example applications.

To begin with, we load the package into Mathematica.

```
In[1]:= << qGeneratingFunctions.m
        qGeneratingFunctions Package by Christoph Koutschan – © RISC Linz – V 1.1 (13.11.2007)
```

2. q-holonomic Sequences and Power Series

Let $\mathbb{K} = \mathbb{Q}(q)$ with q transcendental. For a formal power series $a(x) \in \mathbb{K}[\![x]\!]$, the q-derivative is defined via

$$D_q \sum_{n=0}^{\infty} a_n x^n := \sum_{n=1}^{\infty} a_n \frac{q^n - 1}{q - 1} x^{n-1}.$$

Definition 1. (1) *A sequence* (a_n) *in* $\mathbb{K}$ *is called* q-holonomic *if there exist* $p, p_0, \dots, p_r \in \mathbb{K}[x]$*, not all zero, such that*

$$p_0(q^n)a_n + p_1(q^n)a_{n+1} + \cdots + p_r(q^n)a_{n+r} = p(q^n) \qquad (n \in \mathbb{N}).$$

(2) *A power series* $a(x)$ *over* $\mathbb{K}$ *is called* q-holonomic *if there exist* $p, p_0, \dots, p_r \in \mathbb{K}[x]$*, not all zero, such that*

$$p_0(x)a(x) + p_1(x)D_q a(x) + \cdots + p_r(x)D_q^r a(x) = p(x).$$

A q-holonomic sequence is input by a q-recurrence as above, plus possibly some initial values. If no initial values are given, the results returned by the package will be true for any choice of initial values. In order to define a sequence a_n uniquely, one has to supply r consecutive initial values and the value a_{n+r} at each singular point n, i.e., where $p_r(q^n)$ vanishes. Likewise, a q-holonomic power series is input by a q-differential equation as above, plus possibly some initial values. Again, initial values may be omitted, and in this case generic results will be produced.

It is not essential to Definition 1 that the defining equations may be inhomogeneous, as any inhomogeneous recurrence or differential equation can easily be replaced by an equivalent homogeneous one whenever necessary.

A sequence (a_n) is q-holonomic if and only if its generating function $a(x) = \sum_{n=0}^{\infty} a_n x^n$ is q-holonomic. A specification for $a(x)$ can be computed given a specification for (a_n), and vice versa. Our package provides the commands

$$\text{QRE2DE}\big[\,\textit{spec}\,,\, \textit{fun}_1\,[\,\textit{var}_1\,],\, \textit{fun}_2\,[\,\textit{var}_2\,]\big]$$
$$\text{QDE2RE}\big[\,\textit{spec}\,,\, \textit{fun}_1\,[\,\textit{var}_1\,],\, \textit{fun}_2\,[\,\textit{var}_2\,]\big]$$

for this purpose. The argument *fun₁* [*var₁*] declares the function name and the variable in which the specification of the sequence/power series is given. The argument *fun₂* [*var₂*] declares the function name and the variable which should be used in the output specification:

```
In[2]:= QRE2DE[{a[n] == (q^n + 1)a[n - 1], a[0] == 1}, a[n], A[t]]
Out[2]=        {-(q - 1)qA'[t]t^2 + (-qt - t + 1)A[t] - 1 == 0, A[0] == 1}
In[3]:= QDE2RE[{-(q - 1)A'[t]t^2 + (-qt - t + 1)A[t] - 1 == 0, A[0] == 1}, A[t], a[n]]
Out[3]=        {a[n] == -(-q^n - 1)a[n - 1], a[0] == 1}
```

The q-derivative can be entered with the usual prime (′) for derivatives; this symbol also appears in the output. Internally, however, every prime will be replaced by the symbol QDerivative in order to avoid conflicts with Mathematica's symbol Derivative for the ordinary derivative.

Alternatively, a power series may also be input by a q-shift equation, i.e., an equation involving expressions of the form $f[qx]$, $f[q^2x]$, etc. Initial coefficients are referred to via $\langle 1\rangle[f[x]]$, $\langle x\rangle[f[x]]$, $\langle x^2\rangle[f[x]]$, etc. (i.e., AngleBracket[1][$f[x]$], AngleBracket[x][$f[x]$], AngleBracket[x^2][$f[x]$], etc.). Conversion between q-shift equations and q-differential equations is also supported by the package. In fact, q-derivatives and q-shifts can always be mixed arbitrarily in input. The abbreviations SE and DE are therefore synonymous with respect to input; they only differ with respect to output format.

In[4]:= $\mathbf{QRE2SE[\{a[n] == (q^n + 1)a[n-1], a[0] == 1\}, a[n], A[t]]}$

Out[4]= $\{(1-t)A[t] - qtA[qt] - 1 == 0,\ \langle 1\rangle[A[t]] == 1\}$

In[5]:= $\mathbf{QSE2RE[\{(1-t)A[t] - qtA[qt] - 1 == 0, \langle 1\rangle[A[t]] == 1\}, A[t], a[n]]}$

Out[5]= $\{a[n] == -(-q^n - 1)a[n-1],\ a[0] == 1\}$

In[6]:= $\mathbf{QSE2DE[\{(1-t)A[t] - qtA[qt] - 1 == 0, \langle 1\rangle[A[t]] == 1\}, A[t]]}$

Out[6]= $\{-(q-1)qt^2A'[t] + (-qt - t + 1)A[t] - 1 == 0,\ A[0] == 1\}$

In[7]:= $\mathbf{QDE2SE[\{-(q-1)qt^2A'[t] + (-qt - t + 1)A[t] - 1 == 0, A[0] == 1\}, A[t]]}$

Out[7]= $\{(1-t)A[t] - qtA[qt] - 1 == 0,\ \langle 1\rangle[A[t]] == 1\}$

3. Closure Properties

We say that the class of q-holonomic sequences (or power series) is closed under an operation $\mathcal{F}$ if $\mathcal{F}(a_1, a_2)$ is q-holonomic whenever a_1, a_2 are. Some useful closure properties are discussed in this section. For all of these, our package provides a command for computing a defining equation for $\mathcal{F}(a_1, a_2)$ from defining equations for a_1, a_2. As the proofs are analogous to the ordinary case [21, 11, 8], we will confine ourselves to provide the proof idea.

3.1. Sum.

Theorem 1. (1) *If (a_n) and (b_n) are q-holonomic sequences, then so is $(a_n + b_n)$.*
(2) *If $a(x)$ and $b(x)$ are q-holonomic power series, then so is $a(x) + b(x)$.*

Proof idea. (1) If (a_n) and (b_n) are q-holonomic, they satisfy homogeneous linear recurrence equations, say of orders r_1 and r_2, respectively. Let $c_n := a_n + b_n$. Using the recurrence equations, every term $a_{n+k} + b_{n+k}$ $(k \in \mathbb{N})$ can be rewritten as a $\mathbb{K}(q^n)$-linear combination of a_{n+i} and b_{n+j} for $i = 0, \ldots, r_1 - 1$ and $j = 0, \ldots, r_2 - 1$. It follows that at most $r_1 + r_2$ terms c_{n+k} can be linearly independent. Therefore $c_n, \ldots, c_{n+r_1+r_2}$ are linearly dependent. The dependence is the desired recurrence.
(2) Follows from (1).

□

A specification for the sum of two q-holonomic sequences (power series) can be computed from specifications of the two summands, using one of the following commands:

QREPlus[*spec1* , *spec2* , *fun* [*var*]],
QDEPlus[*spec1* , *spec2* , *fun* [*var*]],
QSEPlus[*spec1* , *spec2* , *fun* [*var*]].

Here, *spec1* and *spec2* are specifications for the summands, both given using *fun* as function symbol and *var* as variable. These symbols will be used again in the output specification of the sum.

Example 1.

In[8]:= $\mathbf{QREPlus[\{a[n] == q^n a[n-1], a[0] == 1\}, \{a[n] == 2a[n-1], a[0] == 1\}, a[n]]}$

Out[8]= $\{(2q - q^n)a[n] == 2q^n(q^n - 2)a[n-2] - (q^{2n} - 4q)a[n-1],\ a[0] == 2,\ a[1] == q + 2\}$

In[9]:= $\mathbf{QDEPlus[\{a'[x] - xa[x] == 0, a[0] == 1\}, \{a'[x] - 2a[x] == 0, a[0] == 1\}, a[x]]}$

Out[9]= $\{2(qx^2 - 2x + 1)a[x] + (3 - qx^2)a'[x] + (x - 2)a''[x] == 0,\ a[0] == 2,\ a'[0] == 2\}$

In[10]:= $\mathbf{QSEPlus[\{(1-x)a[qx] - a[x] == 0, a[0] == 1\}, \{a[qx] - a[x] == 0, a[0] == 1\}, a[x]]}$

Out[10]= $\{-qa[x] + (-xq + q + 1)a[qx] + (qx - 1)a[q^2x] == 0,\ \langle 1\rangle[a[x]] == 2,\ \langle x\rangle[a[x]] == \frac{1}{q-1}\}$

3.2. Hadamard Product.

Theorem 2. (1) *If (a_n) and (b_n) are q-holonomic sequences, then so is their* Hadamard product *(a_nb_n).*

(2) *If $a(x) = \sum_{n=0}^{\infty} a_nx^n$ and $b(x) = \sum_{n=0}^{\infty} b_nx^n$ are q-holonomic power series, then so is their* Hadamard product *$a(x) \odot b(x) := \sum_{n=0}^{\infty} a_nb_nx^n$.*

Proof idea. (1) If (a_n) and (b_n) are q-holonomic, they satisfy homogeneous linear recurrence equations, say of orders r_1 and r_2, respectively. Let $c_n := a_nb_n$. Using the recurrence equations, every term $a_{n+k}b_{n+k}$ $(k \in \mathbb{N})$ can be rewritten as a $\mathbb{K}(q^n)$-linear combination of terms $a_{n+i}b_{n+j}$ for $i = 0, \ldots, r_1 - 1$ and $j = 0, \ldots, r_2 - 1$. It follows that at most r_1r_2 terms c_{n+k} can be linearly independent. Therefore $c_n, \ldots, c_{n+r_1r_2}$ are linearly dependent. The dependence is the desired recurrence.

(2) Follows from (1).

□

A specification for the Hadamard product of two q-holonomic sequences (power series) can be computed from specifications of the two factors, using one of the following commands:

$$\text{QREHadamard}\big[\,\textit{spec}_1\ ,\ \textit{spec}_2\ ,\ \textit{fun}\,[\,\textit{var}\,]\big],$$
$$\text{QDEHadamard}\big[\,\textit{spec}_1\ ,\ \textit{spec}_2\ ,\ \textit{fun}\,[\,\textit{var}\,]\big],$$
$$\text{QSEHadamard}\big[\,\textit{spec}_1\ ,\ \textit{spec}_2\ ,\ \textit{fun}\,[\,\textit{var}\,]\big].$$

Example 2.

In[11]:= $\mathbf{QREHadamard[\{a[n] == q^n a[n-1], a[0] == 1\}, \{a[n] == 2a[n-1], a[0] == 1\}, a[n]]}$

Out[11]= $\{a[n] == 2q^n a[n-1],\ a[0] == 1\}$

In[12]:= $\mathbf{QDEHadamard[\{a'[x] - xa[x] == 0, a[0] == 1\}, \{a'[x] - 2a[x] == 0, a[0] == 1\}, a[x]]}$

Out[12]= $\{-xq^2a^{(3)}[x] + (-q-1)a''[x] + 4a[x] == 0,\ a[0] == 1,\ a'[0] == 0,\ a''[0] == \frac{4}{q+1}\}$

In[13]:= $\mathbf{QSEHadamard[\{(1-x)a[qx] - a[x] == 0, a[0] == 1\},}$
$\mathbf{\{a[qx] - a[x] == 0, a[0] == 1\}, a[x]]}$

Out[13]= $\{a[x] + (qx-2)a[qx] + (1-qx)a[q^2x] == 0,\ \langle 1\rangle[a[x]] == 1, \langle x\rangle[a[x]] == 0\}$

3.3. Cauchy Product.

Theorem 3. (1) *If (a_n) and (b_n) are q-holonomic sequences, then so is their* Cauchy product *$(\sum_{k=0}^{n} a_kb_{n-k})$.*

(2) *If $a(x)$ and $b(x)$ are q-holonomic power series, then so is their* Cauchy product *$a(x)b(x)$.*

Proof idea. (1) Follows from (2).

(2) If $a(x)$ and $b(x)$ are q-holonomic, they satisfy homogeneous linear differential equations, say of orders r_1 and r_2, respectively. Let $c(x) = a(x)b(x)$. Using the differential equations, every term $D_q^kc(x)$ $(k \in \mathbb{N})$ can be rewritten as a $\mathbb{K}(x)$-linear combination of terms $D_q^ia(x)D_q^jb(x)$ for $i = 0, \ldots, r_1 - 1$ and $j = 0, \ldots, r_2 - 1$. It follows that at most r_1r_2 terms $D_q^kc(x)$ can be linearly independent. Therefore $c_n, \ldots, c_{n+r_1r_2}$ are linearly dependent. The dependence is the desired differential equation.

□

A specification for the Cauchy product of two q-holonomic sequences (power series) can be computed from specifications of the two factors, using one of the following commands:

$$\text{QRECauchy}\big[\,\textit{spec}_1\ ,\ \textit{spec}_2\ ,\ \textit{fun}\,[\,\textit{var}\,]\big],$$
$$\text{QDECauchy}\big[\,\textit{spec}_1\ ,\ \textit{spec}_2\ ,\ \textit{fun}\,[\,\textit{var}\,]\big],$$
$$\text{QSECauchy}\big[\,\textit{spec}_1\ ,\ \textit{spec}_2\ ,\ \textit{fun}\,[\,\textit{var}\,]\big].$$

Example 3.

In[14]:= $\boldsymbol{\mathrm{QRECauchy}[\{a[n] == q^n a[n-1], a[0] == 1\}, \{a[n] == 2a[n-1], a[0] == 1\}, a[n]]}$

Out[14]= $\{a[n] == -2q^n a[n-2] - (-q^n - 2)a[n-1],\ a[0] == 1,\ a[1] == q+2\}$

In[15]:= $\boldsymbol{\mathrm{QDECauchy}[\{a'[x] - xa[x] == 0, a[0] == 1\}, \{a'[x] - 2a[x] == 0, a[0] == 1\}, a[x]]}$

Out[15]= $\{(2qx^2 - 2x^2 + x + 2)a[x] - a'[x] == 0,\ a[0] == 1\}$

In[16]:= $\boldsymbol{\mathrm{QSECauchy}[\{(1-x)a[qx] - a[x] == 0, a[0] == 1\},}$
$\boldsymbol{\{a[qx] - (1-qx^2)a[x] == 0, a[0] == q^2\}, a[x]]}$

Out[16]= $\{(1-qx^2)a[x] + (x-1)a[qx] == 0,\ \langle 1\rangle[a[x]] == q^2\}$

3.4. Substitution.

Theorem 4. (1) *If (a_n) is a q-holonomic sequence, then so is $(a_{\alpha n+\beta})$ for any $\alpha \geq 0, \beta \geq 0$.*
(2) *If $a(x)$ is a q-holonomic power series, then so is $a(\lambda x^\alpha)$ for any $\alpha \in \mathbb{N}$ and any $\lambda \in \mathbb{K}$.*

Proof idea. (1) If (a_n) is q-holonomic, it satisfies a homogeneous linear recurrence equation, say of order r. Let $c_n := a_{\alpha n+\beta}$. Using the recurrence equation, every term c_{n+k} $(k \in \mathbb{N})$ can be rewritten as a $\mathbb{K}(q^n)$-linear combination of $a_{n+\beta+i}$ for $i = 0, \ldots, r-1$. It follows that at most r terms c_{n+k} can be linearly independent. Therefore $c_n, \ldots, c_{n+r}$ are linearly dependent. The dependence is the desired recurrence.
(2) If $a(x)$ is q-holonomic, it satisfies a homogeneous linear q-shift equation, say of order r. Using this equation, every term $a(\lambda q^{\alpha k} x)$ $(k \in \mathbb{N})$ can be rewritten as a $\mathbb{K}(x)$-linear combination of $a(\lambda q^i x)$ for $i = 0, \ldots, r-1$. It follows that at most r terms $a(\lambda q^{\alpha k} x)$ can be linearly independent. Therefore, $a(\lambda x), a(\lambda q^\alpha x), \ldots, a(\lambda q^{\alpha r} x)$ are linearly dependent. After the substitution $x \to x^\alpha$, their dependence yields the desired q-shift equation.

□

The class of ordinary holonomic power series is even closed under substitution with algebraic power series [21]. Statement (2) of the preceding theorem corresponds only to a rather limited special case of this. A q-analogue of the more general result is not known.

A specification for the sequence (resp. power series) resulting from substituting $n \mapsto \alpha n + \beta$ (resp. $x \mapsto x^\alpha$) into a sequence (resp. power series) can be computed from a specification of the latter, by one of the following commands:

$$\mathrm{QRESubstitute}\big[\, spec\, , fun\,[\, var\,], \alpha\; var\; + \beta\big],$$
$$\mathrm{QDESubstitute}\big[\, spec\, , fun\,[\, var\,],\; var^{\,\alpha}\big],$$
$$\mathrm{QSESubstitute}\big[\, spec\, , fun\,[\, var\,],\; var^{\,\alpha}\big].$$

Example 4.

In[17]:= $\boldsymbol{\mathrm{QRESubstitute}[\{a[n] = q^n a[n-1] + a[n-2], a[0] == 1, a[1] == q\}, a[n], 2n+3]}$

Out[17]= $\{a[n] == -q^2 a[n-2] - (-q^{4n+5} - q^2 - 1)a[n-1],$
$a[0] == q(q^5+q^2+1),\ a[1] == q(q+1)(q^2+1)(q^2-q+1)(q^4+1)(q^5-q^3+1)\}$

In[18]:= $\boldsymbol{\mathrm{QDESubstitute}[\{a'[x] - (q-x)a[x] == 0, a[0] == 1\}, a[x], x^2]}$

Out[18]= $\{x(-q^3x^6 + q^2x^6 + q^4x^4 - q^2x^4 - q^4x^2 + q^3x^2 + q^2x^2 + x^2 - q^2 - q)a[x] + a'[x] == 0,\ a[0] == 1\}$

In[19]:= $\boldsymbol{\mathrm{QSESubstitute}[\{(1-x)a[qx] - a[x] == 0, a[0] == 1\}, a[x], x^2]}$

Out[19]= $\{(x-1)(x+1)(qx^2-1)a[qx] - a[x] == 0,\ \langle 1\rangle[a[x]] == 1\}$

4. Additional Functions

4.1. **Evaluation.** Given a specification of a q-holonomic sequence or power series, we can compute its first terms (resp. coefficients) $a_0, \dots, a_k$ using the commands

$$\begin{aligned}&\text{QRE2L}\big[\,\textit{spec}\,,\,\textit{fun}\,[\,\textit{var}\,],\ \textit{range}\,\big],\\&\text{QDE2L}\big[\,\textit{spec}\,,\,\textit{fun}\,[\,\textit{var}\,],\ \textit{range}\,\big],\\&\text{QSE2L}\big[\,\textit{spec}\,,\,\textit{fun}\,[\,\textit{var}\,],\ \textit{range}\,\big].\end{aligned}$$

Example 5.

In[20]:= $\boldsymbol{values = \mathrm{QRE2L}[\{a[n+1] == (q^n+1)a[n], a[0] == 1\}, a[n], 9]}$

Out[20]= $$\begin{gathered}\big\{1,\ q+1,\ (q+1)(q^2+1),\ (q+1)^2(q^2+1)(q^2-q+1),\ (q+1)^2(q^2+1)(q^2-q+1)(q^4+1),\\(q+1)^3(q^2+1)(q^2-q+1)(q^4+1)(q^4-q^3+q^2-q+1),\ \dots\\(q+1)^5(q^2+1)^2(q^2-q+1)^2(q^4+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)\\\times(q^6-q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q^8+1)\big\}\end{gathered}$$

In[21]:= $\boldsymbol{\mathrm{QDE2L}[\{a'[x] == (x-q)a[x], a[0] == 1\}, a[x], 4]}$

Out[21]= $$\big\{1, -q, \tfrac{q^2+1}{q+1}, -\tfrac{q(q^2+q+2)}{(q+1)(q^2+q+1)}, \tfrac{2q^4+2q^3+4q^2+q+1}{(q+1)^2(q^2+1)(q^2+q+1)}\big\}$$

In[22]:= $\boldsymbol{\mathrm{QSE2L}[\{a[qx] == (1-x)a[x], a[0] == 1\}, a[x], 4]}$

Out[22]= $$\big\{1, -\tfrac{1}{q-1}, \tfrac{1}{(q-1)^2(q+1)}, -\tfrac{1}{(q-1)^3(q+1)(q^2+q+1)}, \tfrac{1}{(q-1)^4(q+1)^2(q^2+1)(q^2+q+1)}\big\}$$

4.2. **Guessing.** A reverse function is provided which makes it possible to guess a recurrence equation satisfied by a list of values:

$$\text{QREGuess}\big[\,\textit{list}\,,\,\textit{fun}\,[\,\textit{var}\,]\big].$$

We have included this function for the sake of completeness, although there is already more sophisticated guessing machinery for q-sequences available [10, 19].

Example 6.

In[23]:= $\mathbf{QREGuess}[\boldsymbol{values}, \boldsymbol{a[n]}]$

Out[23]= $$\big\{-a[n] == -(1+q^n)a[n-1],\ a[0] == 1\big\}$$

4.3. **Equation Manipulation Tools.** In general, the recurrence or differential equation returned by our commands are inhomogeneous. At the cost of increasing their order, they can be converted into equivalent homogeneous equations by using one of the commands

$$\begin{aligned}&\text{QREHomogeneous}\big[\,\textit{spec}\,,\,\textit{fun}\,[\,\textit{var}\,]\big],\\&\text{QDEHomogeneous}\big[\,\textit{spec}\,,\,\textit{fun}\,[\,\textit{var}\,]\big],\\&\text{QSEHomogeneous}\big[\,\textit{spec}\,,\,\textit{fun}\,[\,\textit{var}\,]\big].\end{aligned}$$

Example 7.

In[24]:= $\boldsymbol{\mathrm{QREHomogeneous}[\{a[n] == (1+q^n)a[n-1] - q^n, a[0] == 1\}, a[n]]}$

Out[24]= $$\big\{a[n] == -(q^n+q)a[n-2] - (-q^n-q-1)a[n-1],\ a[0] == 1,\ a[1] == 1\big\}$$

In[25]:= $\boldsymbol{\mathrm{QDEHomogeneous}[\{a'[x] == xa[x] + 1, a[0] == 1\}, a[x]]}$

Out[25]= $$\big\{-a[x] - qxa'[x] + a''[x] == 0,\ a[0] == 1,\ a'[0] == 1,\ a''[0] == 1\big\}$$

In[26]:= $\boldsymbol{\mathrm{QSEHomogeneous}[\{(1-x)a[qx] == xa[x] + 1, a[0] == 1\}, a[x]]}$

Out[26]= $$\begin{gathered}\big\{xa[x] + (-qx+x-1)a[qx] + (1-qx)a[q^2x] == 0,\\\langle 1\rangle[a[x]] == 1,\ \langle x\rangle[a[x]] == \tfrac{2}{q},\ \langle x^2\rangle[a[x]] == \tfrac{2(q+1)}{q^3}\big\}\end{gathered}$$

5. Applications

5.1. **q-Hypergeometric Identities.** Recurrence equations satisfied by q-hypergeometric sums can be computed with a q-analogue of Zeilberger's algorithm implemented in Mathematica by Paule and Riese [14]. This package in connection with ours allows for proving q-hypergeometric summation identities effortlessly. As an example, consider Bressoud's identity [4]

$$\sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{k=-n}^{n} (-1)^k q^{k(5k+1)/2} \begin{bmatrix} 2n \\ n+2k \end{bmatrix}_q,$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ denotes the q-binomial coefficient. This identity, in connection with the Jacobi triple product identity, implies the Rogers-Ramanujan identities.

We prove this identity by first computing recurrence equations for both sides of the equation.

In[27]:= **<< qZeil.m**

q-Zeilberger Package by Axel Riese – © RISC Linz – V 2.42 (02/18/05)

In[28]:= $\boldsymbol{rec1 = \mathrm{qZeil}[q^{k^2}\mathrm{qBinomial}[n, k, q], \{k, 0, n\}, n]}$

Out[28]= $$\mathrm{SUM}[n] == -\frac{-q^{2n} + q^{n+1} - q^2 - q}{q}\mathrm{SUM}[n-1] - q(1 - q^{n-1})\mathrm{SUM}[n-2]$$

In[29]:= $\boldsymbol{rec2 = \mathrm{qZeil}[(-1)^k q^{k(5k+1)/2}\mathrm{qBinomial}[2n, n+2k, q], \{k, -n, n\}, n]}$

Out[29]= $$\mathrm{SUM}[n] == \frac{q^{2n} + q^{2n+1} + q^{2n+2} + q^7 + q^6 + q^5 + q^4 + q^3}{q^3}\mathrm{SUM}[n-1] + (\ldots\text{messy}\ldots) + (1 - q^{2n-9})(1 - q^{2n-8})q^{10}\mathrm{SUM}[n-5]$$

As we obtain different recurrence equations, the identity is not yet evident at this point. Recurrence equations of lower order can often be obtained by applying clever tricks to the sums in question (such as "creative symmetrizing" [13]), but this requires human expertise, which we prefer to avoid. We continue the proof mechanically by combining the above recurrence equations to a recurrence equation satisfied by the difference of left hand side and right hand side.

In[30]:= $\boldsymbol{\mathrm{QREPlus}[rec1, rec2, \mathrm{SUM}[n]]}$

Out[30]= $$\{q^{12}\mathrm{SUM}[n] == (q^{2n} + q^{2n+1} + q^{2n+2} + q^7 + q^6 + q^5 + q^4 + q^3)q^9\mathrm{SUM}[n-1] + (\ldots\text{messy}\ldots) - (q^4 - q^n)(q^n + q^4)(q^9 - q^{2n})q^5\mathrm{SUM}[n-5]\}$$

As this recurrence has order five, the proof of the identity is completed by checking five initial values, which is easily done. (We may regard it as an irrelevant coincidence that the resulting recurrence is identical to rec2.)

In[31]:= **Table[**
Sum[q^{k^2}qBinomial[n, k, q], {$k, 0, n$}]
== Sum[$(-1)^k q^{k(5k+1)/2}$qBinomial[$2n, n+2k, q$], {$k, -n, n$}],
{$n, 0, 4$}] // qSimplify // FullSimplify

Out[31]= {True, True, True, True, True}

In the same way, finite versions of further identities like the Goellnitz-Gordon identity or the Rogers-Selberg identity [12] can now be done without any human preprocessing.

5.2. **q-Trigonometric Functions.** Two q-analogues of sine and cosine are given by

$$\sin_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(q;q)_{2n+1}}, \qquad \mathrm{Sin}_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2n+1)n} x^{2n+1}}{(q;q)_{2n+1}},$$

$$\cos_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(q;q)_{2n}}, \qquad \mathrm{Cos}_q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(2n-1)n} x^{2n}}{(q;q)_{2n}}.$$

Exercise 1.14 of Gasper/Rahman [6] asks for proving the relations

(1) $\sin_q(x)\,\mathrm{Sin}_q(x) + \cos_q(x)\,\mathrm{Cos}_q(x) = 1$,
(2) $\sin_q(x)\,\mathrm{Cos}_q(x) - \mathrm{Sin}_q(x)\cos_q(x) = 0$.

The exercise can be solved by typing the following commands.

```
In[32]:= sinDe = QRE2DE[{a[n] == -a[n - 2]/((1 - q^n)(1 - q^(n-1))),
                a[0] == 0, a[1] == 1/(1 - q)}, a[n], f[x]];
In[33]:= SinDe = QRE2DE[{a[n] == -q^(2n-3) a[n - 2]/((1 - q^n)(1 - q^(n-1))),
                a[0] == 0, a[1] == 1/(1 - q)}, a[n], f[x]];
In[34]:= cosDe = QRE2DE[{a[n] == -a[n - 2]/((1 - q^n)(1 - q^(n-1))),
                a[0] == 1, a[1] == 0}, a[n], f[x]];
In[35]:= CosDe = QRE2DE[{a[n] == -q^(2n-3) a[n - 2]/((1 - q^n)(1 - q^(n-1))),
                a[0] == 1, a[1] == 0}, a[n], f[x]];
In[36]:= QDEPlus[
            QDECauchy[sinDe, SinDe, f[x]],
            QDECauchy[cosDe, CosDe, f[x]],
            f[x]]
Out[36]=     {(x^2 q^4 + 1)(q - 1)^2 f^(3)[x] + q(q + 1)(q^2 + 1)(q - 1)x f''[x] + (q + 1)(q^2 + 1)f'[x] == 0,
                                   f[0] == 1,  f'[0] == 0,  f''[0] == 0}
```

This proves (1), because the differential equation together with $f'[0] == f''[0] == 0$ implies $f^{(n)}[0] == 0$ for all $n \geq 1$. Likewise, (2) is proven by

```
In[37]:= QDEPlus[
            QDECauchy[sinDe, CosDe, f[x]],
            QDECauchy[{f[x] == -1}, SinDe, cosDe, f[x]],
            f[x]]
Out[37]=     {(x^2 q^4 + 1)(q - 1)^2 f^(3)[x] + q(q + 1)(q^2 + 1)(q - 1)x f''[x] + (q + 1)(q^2 + 1)f'[x] == 0,
                                   f[0] == 0,  f'[0] == 0,  f''[0] == 0}
```

5.3. **q-Fibonacci Numbers.** Andrews et al. [3] have given the identity

$$d_n e_{n+1} - d_{n+1} e_n = (-1)^n q^{\binom{n}{2}} \qquad (n \geq 0)$$

for (d_n), (e_n) defined by

$$\begin{aligned} d_{n+2} &= d_{n+1} + q^n d_n, \qquad & d_0 = 1, d_1 = 0,\\ e_{n+2} &= e_{n+1} + q^n e_n, \qquad & e_0 = 0, e_1 = 1. \end{aligned}$$

For $q = 1$, this identity reduces to Cassini's identity for Fibonacci numbers. We can show the identity with our package, by typing the following commands.

```
In[38]:= defD = {a[n + 2] == a[n + 1] + q^n a[n], a[0] == 1, a[1] == 0};
In[39]:= defE = {a[n + 2] == a[n + 1] + q^n a[n], a[0] == 0, a[1] == 1};
In[40]:= lhs = QREPlus[
            QREHadamard[defD, QRESubstitute[defE, a[n], n + 1], a[n]],
            QREHadamard[{a[n] == -1}, QRESubstitute[defD, a[n], n + 1], defE, a[n]],
            a[n]]
Out[40]=     {q^11 a[n] == -q^(4n) a[n - 4] + (q + 1)(q^n + q^2)q^(n+7) a[n - 2] + q^(2n+7) a[n - 3] + q^11 a[n - 1],
                    a[0] == 1,  a[1] == -1,  a[2] == q,  a[3] == q^2 + q - (q + 1)(q^2 + 1) + 1}
In[41]:= QREPlus[lhs, {a[n] == -q^(n-1) a[n - 1], a[0] == -1}, a[n]]
Out[41]=                                        {a[n] == 0}
```

5.4. **Sequences Arising in the Borwein-Conjecture.** The first Borwein-conjecture [2] states that if A_n, B_n, C_n are polynomials such that

$$\prod_{j=1}^{n}(1 - q^{3j-2})(1 - q^{3j-1}) = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3),$$

then these polynomials have positive coefficients.

The sequences A_n, B_n, C_n are q-holonomic, but it is not clear at first glance how to obtain defining recurrence equations for them using closure properties. In such situations, automated guessing is often useful. For example, a recurrence candidate for the A_n can be found as follows.

In[42]:= $\boldsymbol{polysA = \mathbf{Table}[\mathbf{Expand}[\mathbf{Product}[(1 - q^{3j-2})(1 - q^{3j-1}), \{j, 1, n\}]], \{n, 0, 45\}]}$
$\quad\boldsymbol{/.\ q^{i\_} :\mapsto \mathbf{If}[\mathbf{Mod}[i, 3] == 0, q^{i/3}, 0];}$

In[43]:= $\boldsymbol{\mathbf{QREGuess}[polysA, a[n]]}$

Out[43]= $\$$Failed

By default, the QREGuess command searches for recurrence equations of order at most two with coefficients quadratically depending on q^n. The above result asserts that no such recurrence exists for the sequence in question. A higher order and a higher degree for the coefficients has to be tried.

In[44]:= $\boldsymbol{\mathbf{QREGuess}[polysA, a[n], \mathbf{MaxOrder} \to 4, \mathbf{MaxDegree} \to 6]}$

Out[44]=
$$\begin{gathered}\{-q^{12}a[n] == -(q^{2n} - q^{n+2} + q^{2n+1} + q^{2n+2} + q^5 + q^4 + q^3)q^9 a[n-1]\\ +(q^{4n} - q^{n+6} - q^{n+7} + q^{2n+4} + 2q^{2n+5} + q^{2n+6} - q^{3n+2} - q^{3n+3} + q^{4n+1} + q^{4n+2} + q^{10} + q^9 + q^8)q^5 a[n-2]\\ -(q^{6n} - q^{n+12} + q^{2n+10} - q^{3n+7} - q^{3n+8} + q^{4n+5} - q^{5n+2} + q^{15})a[n-3],\\ a[0] = (q^2+1)(q^2+q+1)\}\end{gathered}$$

Andrews [2] has given a sum representation for the A_n for which the qZeil package delivers the same recurrence, so the recurrence guessed by our package is indeed correct. However, it does not help in proving the conjecture.

5.5. **q-Orthogonal Polynomials.** The q-Hermite polynomials, which are defined by the recurrence equation

$$2xH_n(x|q) = H_{n+1}(x|q) + (1 - q^n)H_{n-1}(x|q), \quad H_0(x|q) = 1,\ H_1(x|q) = 2x,$$

satisfy the identity

$$\frac{H_n(x|q)^2}{(q;q)_n^2} = \sum_{k=0}^{n} \frac{H_{2n-2k}(x|q)}{(q;q)_k (q;q)_{n-k}^2}\,.$$

In order to prove this identity, we first determine a specification for the left hand side.

In[45]:= $\boldsymbol{recH = \{2xh[n] == h[n+1] + (1 - q^n)h[n-1], h[0] == 1, h[1] == 2x\};}$

In[46]:= $\boldsymbol{recFac = \{h[n] == h[n-1]/(1 - q^n), h[0] == 1\};}$

In[47]:= $\boldsymbol{recFac2 = \mathbf{QREHadamard}[recFac, recFac, h[n]];}$

In[48]:= $\boldsymbol{recLHS = \mathbf{QREHadamard}[recH, recH, recFac2, h[n]]}$

Out[48]=
$$\begin{gathered}\{q(q - q^n)(q^n - 1)^2 h[n] == (q - q^n)(q^n + 4x^2 q - q)h[n-1] - (q^n + 4x^2 q - q)qh[n-2] + q^2 h[n-3],\\ h[0] == 1,\ h[1] == \tfrac{4x^2}{(q-1)^2},\ h[2] == \tfrac{(4x^2+q-1)^2}{(q-1)^4(q+1)^2}\}\end{gathered}$$

In a second step we determine a specification for the right hand side.

In[49]:= $\boldsymbol{recH2 = \mathbf{QRESubstitute}[recH, h[n], 2n];}$

In[50]:= $\boldsymbol{recRHS = \mathbf{QRECauchy}[\mathbf{QREHadamard}[recH2, recFac2, h[n]], recFac, h[n]]}$

Out[50]=
$$\begin{gathered}\{-q(q - q^n)(q^n - 1)^2 h[n] == -(q - q^n)(q^n + 4x^2 q - q)h[n-1] + (q^n + 4x^2 q - q)qh[n-2] - q^2 h[n-3],\\ h[0] == 1,\ h[1] == \tfrac{4x^2}{(q-1)^2},\ h[2] == \tfrac{(4x^2+q-1)^2}{(q-1)^4(q+1)^2}\}\end{gathered}$$

As the two specifications agree, the claim follows.

The identity just proven is a special case of the product linearization formula [7, Thm. 3.4]

$$H_m(x|q)H_n(x|q) = \sum_{k=0}^{\min(m,n)} \frac{(q;q)_m (q;q)_n}{(q;q)_k (q;q)_{m-k} (q;q)_{n-k}} H_{m+n-2k}(x|q),$$

which can be also proven automatically in full generality. As this computation is a little more lengthy, we leave it to the reader as an exercise.

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