AUTOMATED PROOFS FOR SOME STIRLING NUMBER IDENTITIES

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Abstract. We present computer-generated proofs of some summation identities for (q-)Stirling and (q-)Eulerian numbers that were obtained by combining a recent summation algorithm for Stirling number identities with a recurrence solver for difference fields.

1. Introduction

In a recent article [5], summation algorithms for a new class of sequences defined by certain types of triangular recurrence equations are given. With these algorithms it is possible to compute recurrences in n and m for sums of the form

$$F(m,n)=\sum_{k=0}^{n} h(m,n,k)S(n,k)$$

where $h(m,n,k)$ is a hypergeometric term and $S(n,k)$ are, e.g., Stirling numbers or Eulerian numbers. Recall that these may be defined via

$$S_1(n,k)=S_1(n-1,k-1)-(n-1)S_1(n-1,k) \qquad S_1(0,k)=\delta_{0,k}, \tag{1}$$

$$S_2(n,k)=S_2(n-1,k-1)+kS_2(n-1,k) \qquad S_2(0,k)=\delta_{0,k}, \tag{2}$$

$$E_1(n,k)=(n-k)E_1(n-1,k-1)+(k+1)E_1(n-1,k) \qquad E_1(0,k)=\delta_{0,k}. \tag{3}$$

The original algorithms exploit hypergeometric creative telescoping [9]. More generally, the algorithms can be extended to work for any sequence $h(m,n,k)$ that can be rephrased in a difference field in which one can solve creative telescoping problems. Since such problems can be solved in Karr's $\Pi\Sigma$-fields [3, 8], we can allow for $h(m,n,k)$ any indefinitely nested sum or product expression, such as (q-)hypergeometric terms, harmonic numbers $H_k=\sum_{i=1}^{k}\frac{1}{i}$, etc. Moreover, $S(n,k)$ may satisfy any triangular recurrence of the form

$$S(n,k)=a_1(n,k)S(n+\alpha,k+\beta)+a_2(n,k)S(n+\gamma,k+\delta) \tag{4}$$

with $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$ and $\begin{vmatrix}\alpha & \gamma\\ \beta & \delta\end{vmatrix}=\pm 1$ and coefficients $a_1(n,k)$ and $a_2(n,k)$ that can be defined by any indefinite nested sum or product over k. In connection with creative telescoping in $\Pi\Sigma$-fields, the algorithms of [5] directly extend to this more general class of summands.

Given a summand $f(m,n,k)=h(m,n,k)S(n,k)$ as specified above and given a finite set of pairs $S\subseteq\mathbb{Z}^2$, the algorithms construct, if possible, expressions $c_{i,j}(m,n)$, free of k, and $g(m,n,k)$ such that the creative telescoping equation

$$\sum_{(i,j)\in S} c_{i,j}(m,n)f(m+i,n+j,k)=g(m,n,k+1)-g(m,n,k) \tag{5}$$

holds and can be independently verified by simple arithmetic.

Summing (5) over the summation range leads to a recurrence relation, not necessarily homogeneous, of the form

$$\sum_{(i,j)\in S} c_{i,j}(m,n)F(m+i,n+j)=d(m,n). \tag{6}$$

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The validity of this recurrence follows, similar to the hypergeometric setting [6], from (5), but is typically not obvious if (5) is not available. Therefore, $g(m,n,k)$ (the only information contained in (5) but not in (6)) is called the *certificate* of the recurrence.

In the following section, we give a detailed example for proving a Stirling number identity involving harmonic numbers in this way. A collection of further identities about q-Stirling numbers that can be proven analogously is given afterwards.

2. A Detailed Example

Consider the sum

$$F(m,n) = \sum_{k=1}^{m} \underbrace{\underbrace{H_{m-k}(m-k)!(-1)^{m-k+1}\binom{m}{k-1}}_{=:h(m,n,k)} \underbrace{S_1(k-1,n)}_{=:S(n,k)}}_{=:f(m,n,k)}.$$

Here, S_1 refers to the (signed) Stirling numbers of the first kind.

The algorithm of [5] reduces the recurrence construction to some creative telescoping problems which can be solved by algorithms for $\Pi\Sigma$ fields [7]. The solutions to all these equations are combined to the recurrence equation

$$\begin{aligned} &F(m,n) - 2mF(m,n+1) - 2F(m+1,n+1)\\ &\quad + m^2F(m,n+2) + (2m+1)F(m+1,n+2) + F(m+2,n+2)\\ &\quad\quad = S_1(m-1,n+1) - (m-1)S_1(m-1,n+2), \end{aligned}$$

which the algorithm returns as output along with the certificate

$$\begin{aligned} g(m,n,k) &= \tfrac{(k-1)}{(k-m-3)(k-m-2)}(-1)^{m-k}(m-k)!\binom{m}{k-1}\\ &\times \big((k^2-3mk-6k+2m^2+6m+6+(k-2)(k-m-1)H_{m-k})S_1(k-1,n+2)\\ &\quad + (k-m-3)((k-m-1)H_{m-k}-1)S_1(k-1,n+1)\big). \end{aligned}$$

The certificate $g(m,n,k)$ allows us to verify the recurrence for $F(m,n)$ independently. Indeed, using the triangular recurrence (1) for S_1 and the obvious relations for factorials, harmonic numbers, etc. it is readily checked that

$$\begin{aligned} &f(m,n,k) - 2mf(m,n+1,k) - 2f(m+1,n+1,k)\\ &\quad + m^2f(m,n+2,k) + (2m+1)f(m+1,n+2,k) + f(m+2,n+2,k)\\ &\quad\quad = g(m,n,k+1) - g(m,n,k). \end{aligned}$$

Now sum this equation for $k = 1,\dots,m-1$. This gives

$$\begin{aligned} &\sum_{k=1}^{m-1} f(m,n,k) - 2m\sum_{k=1}^{m-1} f(m,n+1,k) - 2\sum_{k=1}^{m-1} f(m+1,n+1,k)\\ &\quad + m^2\sum_{k=1}^{m-1} f(m,n+2,k) + (2m+1)\sum_{k=1}^{m-1} f(m+1,n+2,k) + \sum_{k=1}^{m-1} f(m+2,n+2,k)\\ &\quad\quad = \sum_{k=1}^{m-1}\big(g(m,n,k+1) - g(m,n,k)\big). \end{aligned}$$

The right hand side collapses to $g(m,n,m) - g(m,n,1)$. On the left hand side, we can express the sums in terms of the $F(m+i,n+j)$ using, e.g.,

$$\sum_{k=1}^{m-1} f(m+1,n+2,k) = F(m+1,n+2) - f(m+1,n+2,m) - f(m+1,n+2,m+2).$$

Bringing finally everything but the $F(m+i, n+j)$ to the right hand side and doing some straightforward simplifications gives the recurrence claimed by the algorithm.

With the recurrence for $F(m,n)$ at hand, it is an easy matter to prove the closed form representation

$$F(m,n) = \tfrac{1}{2}(n+1)(n+2)S_1(m,n+2).$$

Just check that the closed form satisfies the same recurrence (this is easy) and a suitable set of initial values.

The creative telescoping problems arising during the execution of the algorithm are interesting also from a computational point of view. One of these equations, as an example, is

$$\begin{aligned}&\tfrac{(k-1)(k-m-1)((k-m)H_{m-k}+1)}{k(k-m)^2H_{m-k}}b_2(m,n,k+1) - b_2(m,n,k)\\ &- c_{2,0}(m,n) + \tfrac{(m+1)((m-k+1)H_{m-k}+1)}{(m-k+2)H_{m-k}}c_{2,1}(m,n)\\ &- \tfrac{(m+1)(m+2)((m-k+1)H_{m-k}+1)((m-k+2)H_{m+1-k}+1)}{(m-k+2)(m-k+3)H_{m-k}H_{m+1-k}}c_{2,2}(m,n) = 0,\end{aligned}$$

where $b_2(m,n,k)$ and the $c_i(n,m)$ are to be determined. This equation differs from most equations arising from natural (non-Stirling-) sums in that harmonic number expressions also arise in denominators.

3. Some q-Identities

Subsequently, we consider some q-versions of the well-known identities

$$\sum_{k=m}^{n} \binom{n}{k} S_2(k,m) = S_2(n+1,m+1), \tag{7}$$

$$\sum_{k=m}^{n} (-1)^{n-k} \binom{k}{m} S_1(n,k) = (-1)^{n-m} S_1(n+1,m+1). \tag{8}$$

Following Gould [2], we define the q-Stirling numbers via

$$\begin{aligned}S_1^{(q)}(n,k) &= q^{1-n}S_1^{(q)}(n-1,k-1) - [n-1]S_1^{(q)}(n-1,k), &\quad S_1^{(q)}(0,k) &= \delta_{0,k},\\ S_2^{(q)}(n,k) &= q^{k-1}S_2^{(q)}(n-1,k-1) + [k]S_2^{(q)}(n-1,k), &\quad S_2^{(q)}(0,k) &= \delta_{0,k},\end{aligned}$$

where $[n] = (q^n-1)/(q-1)$ and δ refers to the Kronecker delta. By $\left[{n \atop k}\right]_q$ we denote the q-binomial coefficient, defined as $\left[{n \atop k}\right]_q = [n]!/[k]!/[n-k]!$.

1. We prove the identity [4, Id. 1]

$$\sum_{k=m}^{n} q^k \binom{n}{k} S_2^{(q)}(k,m) = S_2^{(q)}(n+1,m+1)$$

by computing the recurrence

$$q(1-q)F(m+1,n+1) - (1-q)q^{m+2}F(m,n) - q(1-q^{m+2})F(m+1,n) = 0$$

for the sum $F(m,n) = \sum_{k=m}^{n} q^k \binom{n}{k} S_2^{(q)}(k,m)$ with the proof certificate

$$g(m,n,k) = -\frac{k(q-1)q^{k+1}}{k-n-1}\binom{n}{k} S_2^{(q)}(k,m+1).$$

2. The identity [4, Id. 2]

$$\sum_{k=m}^{n} (-1)^{n-k} \binom{k}{m} S_1^{(q)}(n,k) q^{-k} = (-1)^{n-m} S_1^{(q)}(n+1,m+1)$$

follows from the recurrence

$$-(q-1)q^{n+1}F(m+1,n+1) + (q-1)F(m,n) + (q^{n+1}-1)F(m+1,n) = 0$$

with the proof certificate

$$g(m,n,k)=\frac{(-1)^{n-k}(m-k)(q-1)q^{1-k}}{m+1}\binom{k}{m}S_1^{(q)}(n,k-1).$$

3. For the sum

$$F(m,n)=\sum_{k=m}^{n}(-1)^{n-k}\begin{bmatrix}k\\m\end{bmatrix}_q S_1(n,k)q^{-k},$$

involving a q-binomial, we compute the recurrence relation

$$F(m,n)+q(q^m+n)F(m+1,n)-qF(m+1,n+1)=0$$

with the proof certificate

$$g(m,n,k)=-\frac{(-1)^{n-k}q(q^k-q^m)}{q^{m+k}(q^{m+1}-1)}\begin{bmatrix}k\\m\end{bmatrix}_q S_1(n,k-1).$$

We remark that we discovered another q-version of identity (8). Namely, define $\tilde{S}_1^{(q)}(n,k)$ by

$$\tilde{S}_1^{(q)}(n+1,k+1)=q^{-1}\tilde{S}_1^{(q)}(n,k)-(q^k+n)\tilde{S}_1^{(q)}(n,k+1)$$

and $\tilde{S}_1^{(q)}(0,k)=\delta_{0,k}$. Observe that in the limit $q\to 1$ this also specializes to $S_1(n,k)$. Then by construction we get the q-version

$$\sum_{k=m}^{n}(-1)^{n-k}\begin{bmatrix}k\\m\end{bmatrix}_q S_1(n,k)q^{-k}=(-1)^{n-m}\tilde{S}_1^{(q)}(n+1,m+1).$$

4. For

$$F(m,n)=\sum_{k=m}^{n}(-1)^{n-k}\begin{bmatrix}k\\m\end{bmatrix}_q S_1^{(q)}(n,k)q^{-k}$$

we compute the recurrence

$$-(q-1)q^{n+1}F(m+1,n+1)+q(-q^m+q^{m+1}+q^n-1)F(m+1,n)+(q-1)F(m,n)=0$$

with the proof certificate

$$g(m,n,k)=-\frac{(-1)^{n-k}(q-1)q(q^k-q^m)}{q^{m+k}(q^{m+1}-1)}\begin{bmatrix}k\\m\end{bmatrix}_q S_1^{(q)}(n,k-1).$$

If we define $\bar{S}_1^{(q)}(m,n)$ by

$$\bar{S}_1^{(q)}(n+1,k+1)=\frac{1}{(1-q)q^n}(-q^k+q^{k+1}+q^n-1)\bar{S}_1^{(q)}(n+1,k)+q^{-n-1}\bar{S}_1^{(q)}(n,k)$$

and $\bar{S}_1^{(q)}(0,k)=\delta_{0,k}$, which specializes in the limit $q\to 1$ to $S_1(n,k)$, we arrive at the the q-version

$$\sum_{k=m}^{n}(-1)^{n-k}\begin{bmatrix}k\\m\end{bmatrix}_q S_1(n,k)q^{-k}=(-1)^{n-m}\bar{S}_1^{(q)}(n+1,m+1).$$

5. Carlitz [1] defines the q-Eulerian numbers $E_1^{(q)}(n,m)$ by requesting that they satisfy

$$[m]^n=\sum_{k=1}^{n+1}E_1^{(q)}(n,k)\begin{bmatrix}m+k-1\\n\end{bmatrix}_q,$$

which is a q-analogue of the Worpintzky identity [1]. He derives the recurrence equation

$$E_1^{(q)}(n+1,k)=[n+2-k]E_1^{(q)}(n,k-1)+q^{n+1-k}[k]E_1^{(q)}(n,k).$$

Conversely, taking this recurrence equation and suitable initial conditions as the definition of the q-Eulerian numbers, we find that the sum

$$F(n,m)=\sum_{k=1}^{n+1}E_1^{(q)}(n,k)\begin{bmatrix}m+k-1\\n\end{bmatrix}_q$$

satisfies the recurrence

$$(q^m - 1)F(n,m) - (q-1)F(n+1,m) = 0,$$

the certificate being

$$g(m,n,k) = -\frac{q^{-k-1}(q^{k+m}-q)(q^k - q^{n+2})}{q^{n+1}-1}\begin{bmatrix} k+m-2 \\ n \end{bmatrix}_q E_1^{(q)}(n,k-1).$$

The identity $F(m,n) = [m]^n$ follows easily.

Remark. A closed form representation cannot be found for every sum, but almost always it is possible to construct a recurrence equation. For instance, for

$$F(m,n) = \sum_{k=m}^{n} k(-1)^{n-k}\begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n,k)q^{-k}$$

we compute the recurrence relation

$$\begin{aligned} &- q^2(q^{m+1}+n)^2F(m+2,n) + q^2(2q^{m+1}+2n+1)F(m+2,n+1) \\ &\quad - q^2F(m+2,n+2) - q(q^m+q^{m+1}+2n)F(m+1,n) + 2qF(m+1,n+1) - F(m,n) = 0 \end{aligned}$$

with the proof certificate

$$g(m,n,k) = \frac{(-1)^{n-k}q^{-k-2m+1}(q^k-q^m)\begin{bmatrix} k \\ m \end{bmatrix}_q((k-1)(q^{k+1}-1)S_1(n,k-1)q^m + k(q^k-q^{m+1})S_1(n,k-2))}{-q^{m+1}-q^{m+2}+q^{2m+3}+1}.$$

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