Summation Algorithms for Stirling Number Identities

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Abstract

We consider a class of sequences defined by triangular recurrence equations. This class contains Stirling numbers and Eulerian numbers of both kinds, and hypergeometric multiples of those. We give a sufficient criterion for sums over such sequences to obey a recurrence equation, and present algorithms for computing such recurrence equations efficiently. Our algorithms can be used for verifying many known summation identities about Stirling numbers instantly, and also for discovering new identities.

Key words: Symbolic Summation, Stirling Numbers

Find an efficient way to extend
the Gosper-Zeilberger algorithm
from hypergeometric terms
to terms that may involve
Stirling numbers.

Graham, Knuth, Patashnik [4]

1. Introduction

Stirling numbers are interesting not only because of their numerous occurrences in various branches of mathematics, especially in combinatorics, but also because their definition via a triangular recurrence excludes them from all the classes of sequences for which summation algorithms have been devised until now. Summation algorithms are known for hypergeometric summands [3, 16, 15, 9, 14, 10], and, more generally, for holonomic summands [17, 2, 1], for a class of nested sum and product expressions [6, 7, 12, 13], and for several other classes of summands [5, 8, 18, e.g.]. But no algorithm is known by which summation identities about Stirling numbers can be proven.

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In the present paper, we address the research problem of Graham, Knuth and Patashnik quoted above. We define a class of multivariate sequences that contains in particular all the sequences which can be written as the product of a hypergeometric term and a term $S(an+bk, cn+dk)$ referring to the Stirling numbers of first or second kind (a, b, c, d being fixed, specific integers subject to a minor technical restriction). We present a sufficient criterion for sums over such sequences to satisfy a recurrence equation, and we give algorithms that compute a recurrence equation for a given sum efficiently.

With our algorithms, many summation identities about Stirling numbers can be proven effortlessly, by first having the algorithm compute a recurrence for the sum and then checking that the "closed form" of the sum satisfies the same recurrence. There do exist some important Stirling number identities which are not in the scope of our approach. This mainly concerns sums involving expressions of the form n^k which are not hypergeometric with respect to both n and k, and sums with two or more appearances of Stirling number expressions in the summand. We believe that our algorithm could be extended to cover some of the latter, but have no suggestion concerning sums involving expressions of the form n^k.

Our algorithms are implemented as a Mathematica package which is available for download from http://www.risc.uni-linz.ac.at/research/combinat/software/.

2. Preliminaries

2.1. Stirling Numbers

Various different notations for Stirling numbers are used in the literature. We write $S_1(n, k)$, $S_2(n, k)$ for the Stirling numbers of the first and second kind, respectively, and $E_1(n, k)$, $E_2(n, k)$ for the Eulerian numbers of the first and second kind, respectively. These numbers have in common that they may be defined via bivariate "triangular" recurrence equations, as follows:

$$\begin{aligned}
S_1(n,k) &= S_1(n-1,k-1) - (n-1)S_1(n-1,k) & S_1(0,k) &= \delta_{0,k},\\
S_2(n,k) &= S_2(n-1,k-1) + kS_2(n-1,k) & S_2(0,k) &= \delta_{0,k},\\
E_1(n,k) &= (n-k)E_1(n-1,k-1) + (k+1)E_1(n-1,k) & E_1(0,k) &= \delta_{0,k},\\
E_2(n,k) &= (2n+1-k)E_2(n-1,k-1) + (k+1)E_2(n-1,k) & E_2(0,k) &= \delta_{0,k}.
\end{aligned}$$

Motivated by their combinatorial interpretation these numbers are usually only considered for $n, k \geq 0$ and set to 0 outside this range. This, however, implies that the recurrence equations no longer hold on whole $\mathbb{Z}^2$ because of the exceptional point $(n, k) = (0, 0)$. Matters simplify considerably if we set them to 0 only for $n \cdot k < 0$ and otherwise extend the definitions in accordance with the recurrence equations, so that, e.g., $S_2(-3, -4) = 6$.

It should be noted that our definition of S_1 yields the signed Stirling numbers, while some authors prefer to define $|S_1(n, k)|$ as the Stirling numbers of the first kind, and that slightly different definitions for the Eulerian numbers are also in use.

2.2. Operator Algebras

Let C be a field of characteristic zero. The set of bivariate sequences $f : \mathbb{Z}^2 \to C$ together with pointwise addition and multiplication forms a ring. We consider operators of the form

$$\sum_{i,j\in\mathbb{Z}} p_{i,j}(n, k)N^i K^j$$

with $p_{i,j} \in C(n,k)$ at most finitely many of which may be nonzero. These operators act in the usual way on sequences $f\colon \mathbb{Z}^2 \to C$, i.e.,

$$\Big(\sum_{i,j\in\mathbb{Z}} p_{i,j}(n,k)N^iK^j\Big)\cdot f(n,k) = \sum_{i,j\in\mathbb{Z}} p_{i,j}(n,k)f(n+i,k+j) \quad (n,k\in\mathbb{Z}).$$

The set of all operators of the above shape form a noncommutative ring which we denote by $C(n,k)\langle N,K\rangle$. In this ring, we have $Nn = (n+1)N$ and $Kk = (k+1)K$, all other generators commute with each other, e.g. $Nk = kN$. Note that we allow negative powers of N, K, e.g., $N^{-1} + N \in C(n,k)\langle N,K\rangle$.

For a given bivariate sequence $f\colon \mathbb{Z}^2 \to C$, the set

$$\left\{\, Q \in C(n,k)\langle N,K\rangle : Q\cdot f \equiv 0 \,\right\}$$

of all operators that annihilate f forms a left ideal of the ring $C(n,k)\langle N,K\rangle$, called the *annihilator* of the sequence f. For given $Q_1, Q_2, \ldots, Q_l \in C(n,k)\langle N,K\rangle$, we denote by

$$\langle Q_1,\ldots,Q_l\rangle := C(n,k)\langle N,K\rangle Q_1 + \cdots + C(n,k)\langle N,K\rangle Q_l$$

the left ideal generated by $Q_1,\ldots,Q_l$ in $C(n,k)\langle N,K\rangle$. As we will only consider left ideals, we will drop the attribute "left" from now on.

The notation $\mathfrak{a} \trianglelefteq C(n,k)\langle N,K\rangle$ shall indicate that $\mathfrak{a}$ is an ideal in $C(n,k)\langle N,K\rangle$. If $\mathfrak{a}$ is an ideal of $C(n,k)\langle N,K\rangle$ and $p,q \in C(n,k)\langle N,K\rangle$ are such that $p = q + a$ for some $a \in \mathfrak{a}$, then we say that p and q are equivalent modulo $\mathfrak{a}$, written $p \equiv_{\mathfrak{a}} q$.

Classes of sequences may be characterized by restricting the generators of their annihilator to a certain form. For instance, the classical summation algorithms are applicable to (proper) hypergeometric terms [10], which may be defined as follows.

Definition 1. A sequence $f\colon \mathbb{Z}^2 \to C$ is called *hypergeometric* if its annihilator has the form $\langle s_1N - t_1, s_2K - t_2\rangle$ for some $s_1, s_2, t_1, t_2 \in C[n,k]\setminus\{0\}$.

If the s_i, t_i factor into integer-linear factors, then f is called *proper hypergeometric.*

Example 2. The annihilator of the binomial coefficient $\binom{n}{k}$ is

$$\Big\langle\, (k+1)K - (n-k),\ (n-k+1)N - (n+1)\,\Big\rangle \ \trianglelefteq \mathbb{C}[n,k]\langle N,K\rangle.$$

Therefore, $\binom{n}{k}$ is proper hypergeometric.

The terminology and notation introduced above is naturally extended form bivariate sequences to r-variate sequences $\mathbb{Z}^r \to C$ for any fixed $r \in \mathbb{N}$.

3. Stirling-Like Sequences

The class of Stirling-like sequences is defined as the set of all sequences whose annihilators are generated by a triangular recurrence.

Definition 3. A sequence $f\colon \mathbb{Z}^r \to C$ $(r \geq 2)$ is called *Stirling-like* if its annihilator is generated by operators of the form $s_iN_i - t_i$ $(i = 3,\ldots,r)$ for some $s_i, t_i \in C[n_1,\ldots,n_r]\setminus\{0\}$ and an operator of the form

$$u + vN_1^{v_1}N_2^{v_2} - wN_1^{w_1}N_2^{w_2}$$

for some $u, v, w \in C[n_1, \dots, n_r] \setminus \{0\}$ and $v_1, v_2, w_1, w_2 \in \mathbb{Z}$ with $(v_1, v_2)\mathbb{Z} + (w_1, w_2)\mathbb{Z} = \mathbb{Z}^2$ (or, equivalently, $\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = \pm 1$).

If u, v, w and the s_i, t_i factor into integer-linear factors, then f is called *proper Stirling-like.*

We will reflect the distinguished role of the operators N_1, N_2 in the above definition by our choice of naming. As a convention, we will write N, K for the operators playing the roles of N_1, N_2, and use the names $M_1, M_2, \dots$ (or just $M := M_1$) for $N_3, N_4, \dots$. In addition, unless otherwise stated, we will use the following naming conventions:

- C is a field of characteristic zero,
- $r \geq 0$ and $F = C(n, k, m_1, \dots, m_r)$,
- $f \colon \mathbb{Z}^{r+2} \to C$ is a Stirling-like sequence,
- $\mathfrak{a} = \langle u + vN^{v_1}K^{v_2} - wN^{w_1}K^{w_2}, s_1M_1 - t_1, \dots, s_rM_r - t_r \rangle \trianglelefteq F\langle N, K, M_1, \dots, M_r \rangle$ is the annihilator of f,
- $V := N^{v_1}K^{v_2}$, $W := N^{w_1}K^{w_2}$
- We assume without loss of generality that $v_1 \neq 0 \neq w_2$. (This can be done because (v_1, v_2) and (w_1, w_2) are required to generate $\mathbb{Z}^2$, and the roles of V and W may be exchanged if necessary. This assumption will be used in Theorem 19.)
- $\alpha, \beta \in \mathbb{Z}$ shall be such that $\alpha(v_1, v_2) + \beta(w_1, w_2) = (0, 1)$, i.e., $V^\alpha W^\beta = K$. (Again, such a choice is possible because (v_1, v_2) and (w_1, w_2) are required to generate $\mathbb{Z}^2$. These numbers will be used in Sections 5 and 6.)

3.1. Examples and Closure Properties

Example 4. The binomial coefficient $\binom{n}{k}$ is not a Stirling-like sequence, although its annihilator $\mathfrak{a}$ contains $1 + N^{-1} - K$ (Pascal's triangle), which is of the requested form. The reason is that $\langle 1 + N^{-1} - K \rangle \subsetneq \mathfrak{a}$. For example, the operators $(k+1)K - (n-k)$ and $(n-k+1)N - (n+1)$ belong to $\mathfrak{a}$ but not to $\langle 1 + N^{-1} - K \rangle$.

Example 5. The Stirling numbers of the second kind, $S_2(n, k)$, are proper Stirling-like. Owing to the defining recurrence relation the annihilator of S_2 contains $k - N + K^{-1}$. Unlike the binomial coefficient, the sequence S_2 does not satisfy a recurrence pure in K, i.e., the terms $S_2(n, k+i)$ $(i \in \mathbb{Z})$ are linearly independent over $\mathbb{C}[n, k]$. For, suppose otherwise that there are $p_1, \dots, p_r \in \mathbb{C}[n, k]$ with $\gcd(p_1, \dots, p_r) = 1$, not all zero, with

$$p_0(n, k)S_2(n, k) + p_1(n, k)S_2(n, k+1) + \cdots + p_r(n, k)S_2(n, k+r) = 0.$$

Setting $k = 0$ in this recurrence and using the formula

$$S_2(n, i) = \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} j^n (-1)^{i-j}$$

gives

$$\begin{aligned} 0 &= \sum_{i=0}^{r} \frac{p_i(n,0)}{i!} \sum_{j=0}^{i} \binom{i}{j} j^n (-1)^{i-j} = \sum_{i=0}^{r} \sum_{j=0}^{i} \binom{i}{j} \frac{p_i(n,0)}{i!} j^n (-1)^{i-j} \\ &= \sum_{j=0}^{r} \sum_{i=j}^{r} \binom{i}{j} \frac{p_i(n,0)}{i!} j^n (-1)^{i-j} = \sum_{j=0}^{r} \Big(\sum_{i=j}^{r} \binom{i}{j} \frac{p_i(n,0)}{i!} (-1)^{i-j} \Big) j^n. \end{aligned}$$

By the linear independence of the exponential sequences j^n over $\mathbb{C}[n]$ it follows that

$$\sum_{i=j}^{r}\binom{i}{j}\frac{p_i(n,0)}{i!}(-1)^{i-j}=0 \qquad \text{for } j=0,\dots,r \text{ and all } n\geq 0.$$

For $j=r$ follows $p_r(n,0)=0$, then for $j=r-1$ follows $p_{r-1}(n,0)=0$, and successively

$$p_r(n,0)=p_{r-1}(n,0)=\cdots=p_0(n,0)=0 \qquad \text{for all } n.$$

Therefore $k \mid p_i(n,k)$ for $i=0,\dots,r$ in contradiction to $\gcd(p_1,\dots,p_r)=1$.

A similar argument shows that the annihilator of S_2 does not contain a pure recurrence in N. It follows that the annihilator is precisely the ideal $\langle k-N+K^{-1}\rangle$, because any potential annihilating operator of S_2 is equivalent modulo $k-N+K^{-1}$ to an operator pure in N or pure in K.

Also the Stirling numbers of the first kind, $S_1(n,k)$, as well as the Eulerian numbers of first and second kind, $E_1(n,k)$ and $E_2(n,k)$, are Stirling-like with respect to n and k.

Proposition 6. If $a,b,c,d\in\mathbb{Z}$ are such that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1,$$

and $g\colon \mathbb{Z}^{r+2}\to C$ is defined via

$$g(n,k,m_1,\dots,m_r):=f(an+bk,cn+dk,m_1,\dots,m_r),$$

then g is Stirling-like. If f is proper Stirling-like then so is g.

Proof. As shifts in m_i are irrelevant, assume without loss of generality $r=0$. Let $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$ be such that

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

Then $(\alpha,\beta)\mathbb{Z}+(\gamma,\delta)\mathbb{Z}=\mathbb{Z}^2$ and a straightforward calculation confirms that $\langle u+vN^{\alpha}K^{\beta}-wN^{\gamma}K^{\delta}\rangle$ is the annihilator of g. The claim follows. □

Example 7. $S_1(n+k,k)$, $E_2(3n+k,2n+k)$, $E_1(n+23k,k),\dots$ are all proper Stirling-like.

Proposition 8. If $h\colon\mathbb{Z}^{r+2}\to C$ is hypergeometric, and $g\colon\mathbb{Z}^{r+2}\to C$ is defined via

$$g(n,k,m_1,\dots,m_r):=h(n,k,m_1,\dots,m_r)f(n,k,m_1,\dots,m_r),$$

then g is Stirling-like. If f is proper Stirling-like and h is proper hypergeometric, then g is proper Stirling-like.

Proof. Clearly, if $(s_iM_i-t_i)\cdot f=0$ and $(\hat{s}_iM_i-\hat{t}_i)\cdot h=0$ then $(s_i\hat{s}_iM_i-t_i\hat{t}_i)\cdot g=0$ $(i=1,\dots,r)$. Furthermore, if $(u+vN^{v_1}K^{v_2}-wN^{w_1}K^{w_2})\cdot f=0$ and $(s_1N-t_1)\cdot h=(s_2K-t_2)\cdot h=0$, then $(u+vpN^{v_1}K^{v_2}-wqN^{w_1}K^{w_2})\cdot g=0$ for the rational functions

$$p=\frac{h}{N^{v_1}K^{v_2}h} \quad\text{and}\quad q=\frac{h}{N^{w_1}K^{w_2}h}.$$

Clearing denominators leads to an annihilator of the requested form. □

Example 9. $\binom{m}{k} S_1(n+k,k)$, $(k+n)! E_2(3n+k, 2n+k)$, $(-1)^k/(n-k) E_1(n+23k,k)$, $\dots$ are all proper Stirling-like.

Example 10. The term $\binom{m+n}{m+k} S_2(m+k,n)$ is not Stirling-like. However, if we are given the sum

$$f(n,m) = \sum_k \binom{m+n}{m+k} S_2(m+k,n),$$

we can make the substitution $(n,k,m) \mapsto (n, m-n, k+n-m)$ and arrive at the sum

$$f_1(n,m) = \sum_k \binom{m}{k} S_2(k,n),$$

whose summand is proper Stirling-like. With the algorithms described below, a recurrence for f_1 can be computed. Backsubstitution $(n,m) \mapsto (n, n+m)$ leads to a recurrence for the original sum f.

3.2. Normal Forms

We will later be frequently considering operators $Q \in F\langle N, K, M_1, \dots, M_r\rangle$ and their action on the Stirling-like sequence f. It will be convenient to isolate distinguished operators $\bar{Q} \in Q + \mathfrak{a}$ that may be used as "normal forms" of the equivalence class $Q + \mathfrak{a}$.

Lemma 11. Let $Q \in F\langle N, K, M_1, \dots, M_r\rangle$ and

$$d \le \min\{\, i : Q \text{ nontrivially involves a term } V^i W^j \,\}.$$

Then there exist rational functions $a_{i,j}$, almost all of which are zero, such that

$$Q \equiv_{\mathfrak{a}} \sum_{j<0} a_{d,j} V^d W^j + \sum_{i \ge d} a_{i,0} V^i.$$

Proof. First of all, using the relations $M_i \equiv_{\mathfrak{a}} t_i/s_i$ $(i = 1, \dots, r)$, any occurrence of M_i in Q can be eliminated. Therefore, there are rational functions $a_{i,j}$, almost all of which are zero, such that

$$Q \equiv_{\mathfrak{a}} \sum_{i \ge d, j \in \mathbb{Z}} a_{i,j} V^i W^j.$$

Secondly, using the relation $W \equiv_{\mathfrak{a}} \frac{u}{w} + \frac{v}{w} V$, any term $V^i W^j$ can be eliminated at the cost of modifying the coefficients of $V^i W^{j-1}$ and $V^{i+1} W^{j-1}$ appropriately. Note that this operation does not affect the property required for d. Therefore, there are rational functions $a_{i,j}$, almost all of which are zero, such that

$$Q \equiv_{\mathfrak{a}} \sum_{i \ge d, j \le 0} a_{i,j} V^i W^j.$$

Thirdly, using the relation $V \equiv_{\mathfrak{a}} -\frac{u}{v} + \frac{w}{v} W$, any term $V^i W^j$ with $j < 0$ can be eliminated at the cost of modifying the coefficients of $V^{i-1} W^j$ and $V^{i-1} W^{j+1}$ appropriately. This leads to the desired representation. $\square$

The previous lemma says that every operator Q can be brought to a form that resembles the letter Γ in the grid of exponent vectors of V and W. Putting powers of V to the horizontal axis, the vertical part corresponds to the first sum, while the horizontal part corresponds to the second. Because of this interpretation, we refer to the first sum as the "vertical part" and to the second sum as the "horizontal part" of this representation. We show next that this representation is unique.

Lemma 12. Let $Q \in F\langle N, K, M_1, \ldots, M_r\rangle$ and

$$d \leq \min\{\, i : Q \text{ nontrivially involves a term } V^i W^j \,\}.$$

Suppose that

$$Q \equiv_{\mathfrak{a}} \sum_{j<0} a_{d,j} V^d W^j + \sum_{i \geq d} a_{i,0} V^i \text{ and } Q \equiv_{\mathfrak{a}} \sum_{j<0} \hat{a}_{d,j} V^d W^j + \sum_{i \geq d} \hat{a}_{i,0} V^i$$

for some rational functions $a_{i,j}, \hat{a}_{i,j}$, almost all of which are zero. Then $a_{i,j} = \hat{a}_{i,j}$ for all i and j.

Proof. It suffices to show that 0 is the only operator of the shape under consideration which belongs to $\mathfrak{a}$. Indeed, consider an operator

$$B = \sum_{j<0} b_{d,j} V^d W^j + \sum_{i \geq d} b_{i,0} V^i \in \mathfrak{a}.$$

Suppose that $B \neq 0$. Then not all the coefficients in the first sum can be zero, for that would imply a nontrivial relation

$$b_{d,0} V^d + b_{d+1,0} V^{d+1} + \cdots$$

which is inconsistent with the linear independence of powers of V modulo $\mathfrak{a}$. Therefore, we have $b_{d,j} \neq 0$ for at least one $j < 0$.

Using the relation $V \equiv_{\mathfrak{a}} -\frac{u}{v} + \frac{w}{v} W$, the second sum can be brought to the form

$$\sum_{i \geq 0} \bar{b}_{d,i} V^d W^i$$

for some rational functions $\bar{b}_{d,i}$, almost all of which are zero. This implies a nontrivial relation

$$\sum_{j<0} b_{d,j} V^d W^j + \sum_{i \geq 0} \bar{b}_{d,i} V^d W^i,$$

which is inconsistent with the linear independence of powers of W modulo $\mathfrak{a}$. □

4. Existence of a Recurrence

Our goal is to derive an algorithm for computing recurrence equations with polynomial coefficients for sums over Stirling-like terms. As shown by the following example, such a recurrence may fail to exist.

Example 13. Consider the sequence $g(n) := \sum_k (-1)^k \frac{k!}{k+1} S_2(n,k)$. There does not exist a recurrence equation

$$c_0(n)g(n) + c_1(n)g(n+1) + \cdots + c_s(n)g(n+s) = 0$$

of any order s with coefficients $c_i \in \mathbb{C}[n]$ not all zero.

To see this, observe that $g(n) = B_n$, where B_n denotes the nth Bernoulli number, and recall that these numbers do not satisfy a recurrence of the desired type.

We cannot offer an algorithm that would decide for a given Stirling-like term $f(n,k)$ whether or not the sum $g(n) = \sum_k f(n,k)$ satisfies a linear recurrence with polynomial coefficients. Instead, we will show in Theorem 14 below that *all* sums which involve at least one additional free variable satisfy a recurrence equation. This applies to many summation identities for Stirling numbers appearing in the literature. We restrict our attention to those.

The remainder of this section is devoted to proving the existence of recurrence equations for sums over Stirling-like terms with at least two free variables. The proof is an adaption of the corresponding existence proof for proper hypergeometric terms on which Sr. Celine's summation algorithm is based [15, 10] (see also Zimmermann [18] for a far-reaching generalization of Sr. Celine's method). The key property is that every summand sequence admits a recurrence free of the summation variable k. Upon summing over all k, any such recurrence gives rise to a recurrence for the sum.

Theorem 14. If f is proper Stirling-like and $r \geq 1$, then

$$\mathfrak{a} \cap C(n, m_1, \dots, m_r)\langle N, K, M_1, \dots, M_r\rangle \neq \{0\}.$$

In other words: every Stirling-like sequence in three or more variables satisfies a nontrivial recurrence equation whose coefficients are free of k.

It suffices to show that there is a nontrivial relation connecting n, m_1, N, K, M_1, in other words, we may consider the case $r = 1$ without loss of generality. Suppose that $\mathfrak{a} = \langle u + vV - wW, sM - t\rangle$ with s, t, u, v, w factoring into integer-linear factors.

For $p \in C(n,k,m)$ and $i, j, l \in \mathbb{Z}$, let $p^{(i,j,l)}$ be such that $p^{(i,j,l)}V^iW^jM^l = V^iW^jM^lp$ in the ring $C(n,k,m)\langle N, K, M\rangle$, i.e.,

$$p^{(i,j,l)}(n,k,m) := p(n + iv_1 + jw_1, k + iv_2 + jw_2, m + l).$$

Also, for the purpose of this proof, let $\deg_k \frac{p}{q} := \max\{\deg_k p, \deg_k q\}$ for $p, q \in C[n,k,m]$ relatively prime.

Lemma 15. For all $a_l \in C(n,m)$ and for all $L \geq 0$, the term $\sum_{l=0}^{L} a_l M^l$ is equivalent modulo $\mathfrak{a}$ to some $b \in C(n,k,m)$ with $\deg_k b \leq (L+1)\max\{\deg_k s, \deg_k t\}$.

Proof. Using $sM - t \in \mathfrak{a}$ repeatedly, we find that

$$\sum_{l=0}^{L} a_l M^l \equiv_{\mathfrak{a}} \sum_{l=0}^{L} a_l \prod_{\lambda=0}^{l-1} \frac{t^{(0,0,\lambda)}}{s^{(0,0,\lambda)}} = \frac{\sum_{l=0}^{L} a_l \prod_{\lambda=0}^{l-1} t^{(0,0,\lambda)} \prod_{\lambda=l}^{L} s^{(0,0,\lambda)}}{\prod_{\lambda=0}^{L} s^{(0,0,\lambda)}}.$$

Write p, q for the numerator and the denominator, respectively, in the latter expression. Since $\deg_k s^{(0,0,\lambda)} = \deg_k s$ and $\deg_k t^{(0,0,\lambda)} = \deg_k t$ for all $\lambda \in \mathbb{N}$, it follows that

$$\begin{aligned}&\deg_k p \leq (L+1)\max\{\deg_k s, \deg_k t\} \quad \text{and}\\ &\deg_k q \leq (L+1)\deg_k s \leq (L+1)\max\{\deg_k s, \deg_k t\},\end{aligned}$$

as desired. $\square$

Lemma 16. For all $a_{i,j} \in C(n,k,m)$ and for all $I \geq 0$, the term $\sum_{i=0}^{I}\sum_{j=0}^{I-i} a_{i,j}V^iW^j$ is equivalent modulo $\mathfrak{a}$ to $\sum_{i=0}^{I} b_iV^i$ for some $b_i \in C(n,k,m)$ which can be written as finite sums of rational functions $p/q \in C(n,k,m)$ with

$$\deg_k p/q \leq I\max\{\deg_k u, \deg_k v, \deg_k w\} + \max_{i,j}\deg_k(a_{i,j}).$$

Proof. The lemma obviously holds for $I = 0$. Suppose it holds for some $I \geq 0$. We show that it holds for $I+1$. Consider the term $\sum_{i=0}^{(I+1)} \sum_{j=0}^{(I+1)-i} a_{i,j} V^i W^j$. Using $u+vV-wW \in \mathfrak{a}$, we have

$$\begin{aligned}
&\sum_{i=0}^{(I+1)} \sum_{j=0}^{(I+1)-i} a_{i,j} V^i W^j \\
&= \sum_{i=0}^{I} \sum_{j=0}^{I-i} a_{i,j} V^i W^j + \sum_{i=1}^{I+1} a_{i,I+1-i} V^i W^{I+1-i} + a_{0,I+1} W^{I+1} \\
&\equiv_{\mathfrak{a}} \sum_{i=0}^{I} \sum_{j=0}^{I-i} a_{i,j} V^i W^j + \Big(\sum_{i=0}^{I} a_{i+1,I-i} V^i W^{I-i}\Big) V \\
&\qquad + a_{0,I+1} (\tfrac{u}{w})^{(0,I,0)} W^I + a_{0,I+1} (\tfrac{v}{w})^{(1,I,0)} V W^I \\
&= \sum_{i=0}^{I} \sum_{j=0}^{I-i} \bar{a}_{i,j} V^i W^j + \Big(\sum_{i=0}^{I} \sum_{j=0}^{I-i} \hat{a}_{i,j} V^i W^j\Big) V,
\end{aligned}$$

where $\bar{a}_{0,I} = a_{0,I} + a_{0,I+1} (\frac{u}{w})^{(0,I,0)}$ and $\bar{a}_{i,j} = a_{i,j}$ otherwise, and with $\hat{a}_{0,I} = a_{1,I} + a_{0,I+1} (\frac{v}{w})^{(1,I,0)}$ and $\hat{a}_{i,I-i} = a_{i+1,I-i}$ and $\hat{a}_{i,j} = 0$ otherwise. By induction hypothesis, these sums are equivalent modulo $\mathfrak{a}$ to $\sum_{i=0}^{I} \bar{c}_i V^i$ and $\sum_{i=0}^{I} \hat{c}_i V^i$, respectively, where $\bar{c}_i, \hat{c}_i$ are finite sums of rational functions $p/q \in C(n,m,k)$ with

$$\deg_k p/q \leq I \max\{\deg_k u, \deg_k v, \deg_k w\} + \max_{i,j} \deg_k(\bar{a}_{i,j} \text{ [resp. } \hat{a}_{i,j}] \text{)}.$$

We have

$$\sum_{i=0}^{(I+1)} \sum_{j=0}^{(I+1)-j} a_{i,j} V^i W^j \equiv_{\mathfrak{a}} \bar{c}_0 + (\bar{c}_1 + \hat{c}_0)V + \cdots + (\bar{c}_I + \hat{c}_{I-1})V^I + \hat{c}_I V^{I+1}.$$

Since $\deg_k \bar{a}_{i,j}, \deg_k \hat{a}_{i,j} \leq \max\{\deg_k u, \deg_k v, \deg_k w\} + \max_{i,j} \deg_k a_{i,j}$, it follows that the $\bar{c}_i$ and $\hat{c}_i$ are finite sums of rational functions p/q with

$$\deg_k p/q \leq (I+1) \max\{\deg_k u, \deg_k v, \deg_k w\} + \max_{i,j} \deg_k(a_{i,j}),$$

thus the lemma holds for $I+1$. $\square$

Lemma 17. For all $a_{i,j,l} \in C(n,m)$, and for all $I, L \geq 0$, the term

$$\sum_{i=0}^{I} \sum_{j=0}^{I-i} \sum_{l=0}^{L} a_{i,j,l} V^i W^j M^l$$

is equivalent modulo $\mathfrak{a}$ to a term $\sum_{i=0}^{I} b_i V^i$ for certain $b_i \in C(n,k,m)$ which can be written as finite sums of rational functions $p/q \in C(n,k,m)$ with $\deg_k p/q = \mathrm{O}(I+L)$.

Proof. By Lemma 15, the term $\sum_{i=0}^{I} \sum_{j=0}^{I-i} \sum_{l=0}^{L} a_{i,j,l} V^i W^j M^l$ is equivalent to a term $\sum_{i=0}^{I} \sum_{j=0}^{I-i} b_{i,j} V^i W^j$ with certain $b_{i,j} \in C(n,k,m)$ with $\deg_k b_{i,j} = \mathrm{O}(L)$. Now apply Lemma 16. $\square$

Lemma 18. For all $a_{i,j,l} \in C(n,m)$, and for all $I, L \geq 0$, the term

$$\sum_{i=0}^{I} \sum_{j=0}^{I-i} \sum_{l=0}^{L} a_{i,j,l} V^i W^j M^l$$

is equivalent modulo $\mathfrak{a}$ to

$$\frac{c_0}{d} + \frac{c_1}{d} V + \cdots + \frac{c_I}{d} V^I$$

with polynomials $c_i, d \in C[n,k,m]$ with $\deg_k d = \mathrm{O}(I+L)$ and $\deg_k c_i = \mathrm{O}(I+L)$.

Proof. By Lemma 17, the term in question is equivalent to $\sum_{i=0}^{I} b_i V^i$ for certain $b_i \in C(n,k,m)$ which can be written as finite sums of rational functions $p/q \in C(n,k,m)$ with $\deg_k p/q = \mathrm{O}(I+L)$. To show the present lemma, it suffices to show that all these p/q in all the c_i share a common denominator d which satisfies the desired degree bound.

To see this, first observe that following the calculations for Lemma 15 and 16 a common denominator (not necessarily the least) is given by

$$\begin{aligned} d := &\operatorname{lcm}\{\, s^{(i,j,l)} : 0 \leq i \leq I, 0 \leq j \leq I-i, 0 \leq l \leq L \,\} \\ &\cdot \operatorname{lcm}\{\, w^{(i,j,0)} : 0 \leq i \leq I, 0 \leq j \leq I-i \,\}. \end{aligned}$$

For a polynomial $p \in C[n,k,m]$ and $J \in \mathbb{N}$, define the new polynomial

$$\pi(J;p) := (p-J)(p-J+1)\cdots(p-1)p(p+1)\cdots(p+J) \in C[n,k,m].$$

Clearly $\deg_k \pi(J;p) = (2J+1)\deg_k p = \mathrm{O}(J)$. Consider the polynomial

$$D := \prod_{\substack{a,b,c\in\mathbb{Z},h\in C\\ an+bk+cm+h \mid ws}} \pi(|a|(|v_1|+|w_1|)I + |b|(|v_2|+|w_2|)I + |c|L; an+bk+cm+h).$$

By assumption, w and s factor into integer-linear factors, so this product ranges over all irreducible factors of w and s, and a,b,c,h are independent of I and L. As a consequence, we have $d \mid D^e$ for some fixed $e \in \mathbb{Z}$ that bounds the multiplicities of the irreducible factors in w and s. The proof is completed by observing that $\deg_k D^e = \mathrm{O}(I+L)$. □

Proof of Theorem 14. Consider an ansatz $\sum_{i=0}^{I}\sum_{j=0}^{I-i}\sum_{l=0}^{L} a_{i,j,l} V^i W^j M^l$ with undetermined coefficients $a_{i,j,l}$. According to Lemma 18, the ansatz is equivalent modulo $\mathfrak{a}$ to

$$\frac{c_0}{d} + \frac{c_1}{d} V + \cdots + \frac{c_I}{d} V^I$$

for certain c_i, d of degree at most $\mathrm{O}(I+L)$ with respect to k. These c_i, d depend linearly on the $a_{i,j,l}$. Comparing the coefficients with respect to powers of k and V of the numerator to zero thus gives a system of linear equations for the $a_{i,j,l}$ that can be solved in $C(n,m)$. This system has $\frac{1}{2}(L+1)(I+1)(I+2) = \mathrm{O}(I^2 L)$ variables but only $(I+1)\mathrm{O}(I+L) = \mathrm{O}(I^2 + IL)$ equations, hence for sufficiently large I, L the system will have a nontrivial solution. □

5. Indefinite Summation

Theorem 14 gives rise to an algorithm for computing a recurrence equation for a given sum $\sum_k f(n,k,m)$ over a proper Stirling-like sequence with finite support: Guess $I, L \geq 0$

and make an ansatz

$$\sum_{i=0}^{I}\sum_{j=0}^{I-i}\sum_{l=0}^{L} a_{i,j,l}V^iW^jM^l$$

for an annihilating operator of the summand f. Bring the ansatz to normal form (in the sense of Lemma 18), clear denominators, and compare coefficients with respect to powers of V and k. This gives a linear system for the coefficients $a_{i,j,l}$ over $C(n,m)$ which will have a nontrivial solution as soon as I, L are sufficiently large. Any nontrivial solution gives rise to a k-free recurrence for f which in turn gives rise to a recurrence for the given sum.

The algorithm just described is a generalization of Sr. Celine's hypergeometric summation algorithm [10] (and is also close to Zimmermann's generalization of Sr. Celine's algorithm [18]). The drawback of this algorithm is that it may require solving large linear systems with rational function coefficients which is impractical already for moderate examples. Assuming for simplicity $L = I$, we need to determine $\frac{1}{2}(I+1)(I+2)(I+1)$ coefficients $a_{i,j,l}$, which requires roughly $\mathrm{O}(I^9)$ operations in $C(n,m)$ (counting a cubic complexity for linear system solving). If we assume that I is not known a priori, the algorithm has to be applied for $I = 0, 1, 2, \dots$ until a solution is found. This leads to a total complexity of roughly $\mathrm{O}(I^{10})$ field operations.

For the hypergeometric case, Zeilberger's algorithm [10] provides an efficient alternative to Sr. Celine's algorithm. This algorithm is based on Gosper's algorithm for indefinite summation of hypergeometric terms. In the next section, we present a fast algorithm for summation of Stirling-like sequences which is based on Zeilberger's approach. In the present section, we give a procedure for indefinite summation which will later serve as a substitute for Gosper's algorithm.

An operator P is called *summable* if there exists another operator Q with $(K-1)Q \equiv_{\mathfrak{a}} P$. In this event, Q is called an anti-difference of P.

Theorem 19. Recall the convention that $\alpha, \beta \in \mathbb{Z}$ be such that $\alpha(v_1, v_2) + \beta(w_1, w_2) = (0, 1)$, i.e., $K = V^\alpha W^\beta$. Consider

$$P := a_{p_1}V^{p_1} + a_{p_1+1}V^{p_1+1} + \cdots + a_{p_2}V^{p_2}$$

with $p_1 < p_2$, $a_i \in F$, $a_{p_1} \neq 0 \neq a_{p_2}$. If there exists some $Q \in F\langle N, K, M_1, \dots, M_r\rangle$ with

$$(K-1)Q \equiv_{\mathfrak{a}} P,$$

then there also exists a Q with this property which has the form

$$Q = b_{q_1}V^{q_1} + \cdots + b_{q_2}V^{q_2}$$

where $b_j \in F$, $b_{q_1} \neq 0 \neq b_{q_2}$ and

(1) if $\alpha \neq 0$ then $q_1 = p_1 - \min(0, \alpha)$,
(2) if $\alpha + \beta \neq 0$ then $q_2 = p_2 - \max(0, \alpha + \beta)$.

Proof. By Lemma 11, we may assume without loss of generality that

$$Q = \sum_{j<0} b_{d,j}V^dW^j + \sum_{i \geq d} b_{i,0}V^i$$

for some $d \in \mathbb{Z}$ (chosen small enough to work for all normal forms considered in this proof) and some rational functions $b_{i,j}$, almost all of which are zero. We show that the first sum (the "vertical part") is zero. Suppose otherwise, and let $j_0 < 0$ be minimal with $b_{d,j_0} \neq 0$. It follows from $v_1 \neq 0 \neq w_2$ that $\beta \neq 0$. Therefore, the vertical part of $KQ = V^\alpha W^\beta Q$ will start at $j_0 + \beta \neq j_0$. As a consequence, the normal form (in the sense of Lemma 11) of $(K-1)Q$ will have a nonzero vertical part starting at $\min(j_0, j_0+\beta) < 0$. However, by Lemma 12, this normal form is unique, and this is in contradiction to $(K-1)Q \equiv_{\mathfrak{a}} P$, since P is of the form of Lemma 11 but P's vertical part is zero.

Suppose now that $\alpha \neq 0$. Let $q_1 \in \mathbb{Z}$ be minimal such that the coefficient of V^{q_1} in Q is nonzero. Then, if $KQ = V^\alpha W^\beta Q$ is brought to the form of Lemma 11, $q_1 + \alpha$ is the minimal exponent of V with a nonzero coefficient. Since $q_1 + \alpha \neq q_1$, it follows that $\min(q_1+\alpha, q_1) = q_1 + \min(0,\alpha)$ is the minimal exponent of V with nonzero coefficient in $(K-1)Q$. As this exponent must agree with the minimal exponent p_1 of P, it follows that $p_1 = q_1 + \min(0,\alpha)$, thus $q_1 = p_1 - \min(0,\alpha)$, as claimed.

The result for $\alpha + \beta \neq 0$ is shown by a similar argument. $\square$

Corollary 20. For any $Q \in F\langle N, K, M_1, \ldots, M_r\rangle$, we have: if $Q \notin \mathfrak{a}$ then $(K-1)Q \notin \mathfrak{a}$.

Proof. We may assume without loss of generality that Q is in the form of Lemma 11. Like in the proof of Theorem 19, it can be shown that if Q has a nontrivial vertical part, then the normal form of $(K-1)Q$ will have a nontrivial vertical part, too, so $(K-1)Q \notin \mathfrak{a}$ in this case. If Q has no vertical part, then it can be shown like in the proof of Theorem 19 that the normal form of $(K-1)Q$ has no vertical part either, but has at least one nonzero coefficient, so $(K-1)Q \notin \mathfrak{a}$ also in this case. $\square$

Since $\alpha(v_1, v_2) + \beta(w_1, w_2) = (0,1)$, we cannot have $\alpha = \beta = 0$, it is therefore also not possible to have $\alpha = 0$ and $\alpha + \beta = 0$ at the same time. It follows that at least one of the two situations in Theorem 19 ($\alpha \neq 0$ or $\alpha + \beta \neq 0$) occurs. In the fortunate case where $\alpha \neq 0$ *and* $\alpha + \beta \neq 0$, we obtain an upper and lower bound for the range of nonzero coefficients in Q. We can then make an ansatz and solve for the coefficients b_i.

In the less fortunate (but more frequent) case where one of the conditions is satisfied but the other is not, the theorem provides only a one-sided bound for the range of nonzero coefficients. We can still make an ansatz and successively solve for the coefficients b_i of Q, as illustrated in the following examples.

Example 21. Consider the sum $\sum_{k=0}^m \frac{(-1)^k}{k!} S_1(k,n)$. The sequence

$$f\colon \mathbb{Z}^2 \to \mathbb{C}, \quad f(n,k) = \frac{(-1)^k}{k!} S_1(k,n)$$

has the annihilator $\langle k - N^{-1} - (k+1)K\rangle$ and is therefore proper Stirling-like. We have $\alpha = 0, \beta = 1$ and set $P = 1$, thus, if the sequence is summable, the anti-difference will have the form $Q \cdot f$ for some

$$\begin{aligned} Q &= \cdots + b_{-3}(N^{-1})^{-3} + b_{-2}(N^{-1})^{-2} + b_{-1}(N^{-1})^{-1} \\ &= b_{-1}N + b_{-2}N^2 + b_{-3}N^3 + \cdots \end{aligned}$$

with $b_i \in \mathbb{C}(n,k)$ to be determined. We have

$$\begin{aligned} (K-1)Q &= (K-1)(b_{-1}N + b_{-2}N^2 + b_{-3}N^3 + \cdots) \\ &= \big(b'_{-1}NK + b'_{-2}N^2K + b'_{-3}N^3K + \cdots\big) - \big(b_{-1}N + b_{-2}N^2 + b_{-3}N^3 + \cdots\big) \\ &\equiv_{\mathfrak{a}} -\tfrac{1}{k+1}b'_{-1} + (\tfrac{k}{k+1}b'_{-1} - b_{-1} - \tfrac{1}{k+1}b'_{-2})N \\ &\qquad + (\tfrac{k}{k+1}b'_{-2} - b_{-2} - \tfrac{1}{k+1}b'_{-3})N^2 + \cdots \end{aligned}$$

for b_i' such that $b_i'K = Kb_i$. Comparing the coefficients of N^i to $P = 1 + 0N + 0N^2 + \cdots$ gives

$$\begin{aligned} & -\tfrac{1}{k+1}b'_{-1} \stackrel{!}{=} 1 && \Rightarrow \quad b'_{-1} = -k-1 && \Rightarrow \quad b_{-1} = -k \\ \Rightarrow \quad & \tfrac{k}{k+1}(-k-1) - (-k) - \tfrac{1}{k+1}b'_{-2} \stackrel{!}{=} 0 && \Rightarrow \quad b'_{-2} = 0 && \Rightarrow \quad b_{-2} = 0 \end{aligned}$$

Continuation gives $b_{-3} = b_{-4} = \cdots = 0$, so we obtain the solution $Q = -kN$. It follows that f is summable. We have

$$(-k-1)f(n+1, k+1) - (-k)f(n+1, k) = f(n, k),$$

and $\sum_{k=0}^{m} \frac{(-1)^k}{k!} S_1(k, n) = (-m-1)\frac{(-1)^{m+1}}{(m+1)!} S_1(m+1, n+1) = \frac{(-1)^m}{m!} S_1(m+1, n+1)$.

The algorithm terminates as soon as $|\alpha|+|\beta|$ consecutive b_i can be set to zero, because then all further coefficients of Q can be set to zero as well, and the final operator Q consists just of those terms that have been constructed up to this point. It might be that in the course of generating Q, no coefficient ever evaluates to 0, i.e., the algorithm does not terminate. This happens if and only if P is not indefinitely summable.

Example 22. Consider the sum $\sum_{k=0}^{m} \binom{m}{k} S_2(k, n)$. The sequence

$$f\colon \mathbb{Z}^3 \to \mathbb{C}, \quad f(n, k, m) = \binom{m}{k} S_2(k, n)$$

has the annihilator $\mathfrak{a} = \langle (k-m)n + (k-m)N^{-1} + (k+1)K,\ (1-k+m)M - (m+1) \rangle$ and is therefore proper Stirling-like. We have $\alpha = 0, \beta = 1$ and set $P = 1$, thus, if the sequence is summable, the anti-difference will have the form $Q \cdot f$ for some

$$\begin{aligned} Q &= \cdots + b_{-3}(N^{-1})^{-3} + b_{-2}(N^{-1})^{-2} + b_{-1}(N^{-1})^{-1} \\ &= b_{-1}N + b_{-2}N^2 + b_{-3}N^3 + \cdots \end{aligned}$$

with $b_i \in \mathbb{C}(n, k, m)$ to be determined. We have

$$\begin{aligned} (K-1)Q &= (K-1)(b_{-1}N + b_{-2}N^2 + b_{-3}N^3 + \cdots) \\ &= \big(b'_{-1}NK + b'_{-2}N^2K + b'_{-3}N^3K + \cdots\big) - \big(b_{-1}N + b_{-2}N^2 + b_{-3}N^3 + \cdots\big) \\ &\equiv_{\mathfrak{a}} \tfrac{m-k}{k+1}b'_{-1} + \big(\tfrac{(m-k)(n+1)}{k+1}b'_{-1} - b_{-1} + \tfrac{m-k}{k+1}b'_{-2}\big)N \\ &\qquad + \big(\tfrac{(m-k)(n+2)}{k+1}b'_{-2} - b_{-2} + \tfrac{m-k}{k+1}b'_{-3}\big)N^2 + \cdots \end{aligned}$$

for b_i' such that $b_i'K = Kb_i$. Comparing coefficients of N^i with $P = 1$ gives

$$\begin{aligned} & \tfrac{m-k}{k+1}b'_{-1} \stackrel{!}{=} 1 && \Rightarrow \quad b_{-1} = \tfrac{k}{m-k+1} \\ \Rightarrow \quad & \tfrac{(m-k)(n+1)}{k+1}b'_{-1} - b_{-1} + \tfrac{m-k}{k+1}b'_{-2} \stackrel{!}{=} 0 && \Rightarrow \quad b_{-2} = \tfrac{k(nk+2k-m-mn-2n-3)}{(k-m-2)(k-m-1)} \\ \Rightarrow \quad & \tfrac{(m-k)(n+2)}{k+1}b'_{-2} - b_{-2} + \tfrac{m-k}{k+1}b'_{-3} \stackrel{!}{=} 0 && \Rightarrow \quad b_{-3} = -\tfrac{k(n^2k^2+5nk^2+6k^2+\cdots+24n+23)}{(k-m-3)(k-m-2)(k-m-1)} \\ \Rightarrow \quad & \tfrac{(m-k)(n+3)}{k+1}b'_{-3} - b_{-3} + \tfrac{m-k}{k+1}b'_{-4} \stackrel{!}{=} 0 && \Rightarrow \quad b_{-4} = \cdots \end{aligned}$$

The sequence of b_i thus obtained is presumably nonzero throughout, which would mean that f is not indefinitely summable.

Only in the next section we will be able to show the identity $\sum_{k=0}^{m}\binom{m}{k}S_2(k,n) = S_2(m+1,n+1)$, which cannot be done by indefinite summation only. This is analogous to the fact that Gosper's algorithm does not suffice to prove the binomial theorem $\sum_{k=0}^{m}\binom{m}{k} = 2^m$, for instance. A noteworthy difference between Gosper's algorithm and the indefinite summation procedure outlined above is that our procedure terminates if and only if an anti-difference Q actually exists (except when $\alpha \neq 0$ and $\alpha+\beta \neq 0$, where it always terminates), whereas Gosper's algorithm is able to detect also the non-existence of anti-differences. It will turn out in the next section, however, that our procedure is sufficient for our purpose.

6. Definite Summation

Given a hypergeometric term $f\colon \mathbb{Z} \to C$, Gosper's algorithm may find out that f is not indefinitely summable, in the sense that there does not exist a hypergeometric term $g\colon \mathbb{Z} \to C$ such that $g(k+1) - g(k) = f(k+1)$, or, equivalently, there does not exist any $Q \in C(k)\langle K\rangle$ such that $(K-1)Qf = f$. More generally, given s hypergeometric terms $f_0,\dots,f_s\colon \mathbb{Z} \to C$, an extension of Gosper's algorithm can be used in order to find all constants $c_0,\dots,c_s$ such that the linear combination $c_0f_0 + \dots + c_sf_s$ becomes indefinitely summable. The specification of the extended Gosper algorithm is summarized in the following Lemma.

Lemma 23. Let $a_0,a_1,f_1,\dots,f_s \in C(k)$. The set of all tuples $(c_1,\dots,c_s;g) \in C^s \times C(k)$ with

$$(a_1K - a_0)g = c_1f_1 + \dots + c_sf_s$$

forms a finite dimensional vector space over C, say of dimension d, and there is an algorithm which takes a_0,a_1 and $f_1,\dots,f_s$ as input and returns a vector $b \in C(k)^d$ and a matrix $A \in C^{d\times s}$ such that the rows of the augmented matrix $(A|b)$ are linearly independent over C and $(c_1,\dots,c_s;g)$ belongs to the solution space if and only if there exist $e \in C^d$ such that $(c_1,\dots,c_s;g) = e\cdot(A|b)$.

Zeilberger's algorithm for finding recurrence equations satisfied by definite sums is based on the extended Gosper algorithm. If $f\colon \mathbb{Z}^2 \to C$ is a hypergeometric term, then, according to Zeilberger, we apply the extended Gosper algorithm to find $c_0,\dots,c_s \in C(n)$ such that

$$c_0f + c_1[N\cdot f] + \dots + c_s[N^s\cdot f]$$

is indefinitely summable. For, this would imply the existence of $Q \in C(n,k)\langle N,K\rangle$ and $P = c_0 + c_1N + \dots + c_sN^s \in C(n)\langle N\rangle$ such that $(K-1)Q\cdot f = P\cdot f$. If f is such that $\lim_{|k|\to\infty} f(n,k) = 0$ for each fixed $n \in \mathbb{Z}$, then summing this equation over all k makes the left hand side collapse to 0 and so reveals that P annihilates $\sum_k f(n,k)$.

The algorithm is backed by a theorem [10, Thm. 6.2.1] which guarantees that appropriate operators P and Q exist whenever the summand f is proper hypergeometric. The following theorem contains the analogous result for proper Stirling-like sequences.

Theorem 24. Assume that f is proper Stirling-like. Then there exist operators

$$P \in \bigoplus_{i=0}^{|\beta|-1} C(n,m)\langle V,M\rangle W^i \quad\text{and}\quad Q \in F\langle N,K,M\rangle$$

with $(K-1)Q \equiv_{\mathfrak{a}} P$ and $P \neq 0$.

Proof. By Theorem 14, $\mathfrak{a}$ contains an operator $R \neq 0$ which is free of k, i.e., $R \in C(n,m)\langle N,K,M\rangle$. We have $K = V^\alpha W^\beta$ and also $N = V^\gamma W^\delta$ for some $\gamma,\delta \in \mathbb{Z}$ such that $\gamma(v_1,v_2)+\delta(w_1,w_2) = (1,0)$. Thus we may regard R as an element of $C(n,m)\langle V,W,M\rangle = C(n,m)\langle V,W^{-1},M\rangle$. We may assume that R involves only nonnegative powers of $W_+ := W^{\operatorname{sgn}\beta}$ and that the degree with respect to W_+ is minimal among all the R with nonnegative powers of W_+ only. (We can replace R by $W^d R$ for suitable d to achieve this situation.)

Using division with remainder in $C(n,m)\langle V,M\rangle[W_+]$, we can write

$$R = (V^\alpha W_+^{|\beta|} - 1)Q - P$$

for certain $Q \in C(n,k,m)\langle V,W_+,M\rangle$ and $P \in C(n,m)\langle V,W_+,M\rangle$ where the powers of W_+ in P range between 0 and $|\beta|-1$.

It remains to show that $P \neq 0$. Suppose otherwise that $P = 0$. Then $R = (K-1)Q \in \mathfrak{a}$. But then $Q \neq 0$ (because $R \neq 0$) and $Q \in \mathfrak{a}$ (by Cor. 20) and Q is k-free (because R and $V^\alpha W_+^{|\beta|}-1$ are). Since $K = V^\alpha W_+^{|\beta|}$, it follows $0 \leq \deg_{W_+} Q < \deg_{W_+} R$, in contradiction to the minimality assumption on R. □

In the first place, the following three situations have to be distinguished:

Case	Condition	Example Operator
a)	$\alpha = 0$	$u + vN^{-1} - wK$
b)	$\alpha + \beta = 0$	$u + vN - wNK$
c)	$\alpha \neq 0$ and $\alpha + \beta \neq 0$	$u + vN - wN^2K^{-1}$

Note that the cases are mutually exclusive.

We present two algorithms, the first being applicable to all three cases, the second being faster but only applicable to cases a) and b). (Note that the third case is only of little relevance anyway.)

In fact, case b) need not be considered because it can be reduced to case a) as follows. Suppose that $u+vV-wW$ is a generator of case b), i.e., $V = N^{v_1}K^{v_2}$ and $W = N^{w_1}K^{w_2}$ are such that $K = V^\alpha W^\beta$ for some $\alpha,\beta \in \mathbb{Z}$ with $\alpha+\beta = 0$. Then $\alpha(v_1 - w_1) = 0$, which implies $v_1 = w_1$. As (v_1,v_2) and (w_1,w_2) are linearly independent, we must have $v_2 \neq w_2$. Now observe that

$$N^{-v_1}K^{-v_2}(u + vV - wW) = v^{(-1,0,0)} + u^{(-1,0,0)}N^{-v_1}K^{-v_2} - w^{(-1,0,0)}K^{w_2-v_2},$$

which is of case a).

In case a), the triangular generator will always be of the form $u + vV - wK^{\pm1}$, which can be brought to the form $u + vV - wK$ by multiplying by K if necessary. We will proceed to describe our algorithms for this situation. For the modification of the first algorithm that covers case c), an example will suffice.

We continue the use of the symbols $C, F, f, \mathfrak{a}, \ldots$ as introduced in Section 3, and assume in addition $W = K$, $\alpha = 0, \beta = 1$. For notational simplicity, we will also assume $r = 1$. Our goal is to compute an operator $P \in C(n,m)\langle V,M\rangle$ for which there exists a $Q \in C(n,k,m)\langle V\rangle$ with $(K-1)Q \equiv_{\mathfrak{a}} P$. If $Q = \sum_i b_i V^i$ as predicted by Theorem 19, then

$$(K-1)Q \equiv_{\mathfrak{a}} \sum_i ((\tfrac{u}{w})^{(i,0,0)}b_i' - b_i + (\tfrac{v}{w})^{(i-1,0,0)}b_{i-1}')V^i.$$

(Recall that b' was defined such that $b'K = Kb$ for $b \in F$). In indefinite summation (Section 5), we determine the coefficients b_i such as to match the above representation of $(K-1)Q$ to a given operator P (which is 1 for indefinite summation). For a fixed P, this may or may not lead to an operator Q with only finitely many nonzero coefficients b_i. In definite summation, we simultaneously construct both P and Q in such a way that P will eventually be summable, i.e., a finite anti-difference Q exists.

The operator P may be constructed starting with $P = 1$ and adding either increasing or decreasing powers of V with appropriate coefficients. These two possibilities (increasing vs. decreasing powers of V) correspond to the two algorithms given next.

6.1. First Algorithm

In the first algorithm, we extend the ansatz for P with negative powers of V, i.e., we apply the indefinite summation procedure of the previous section to the operator

$$P = \sum_{i \le 0} \sum_{j=0}^{|i|} c_{i,j} V^i M^j$$

with undetermined $c_{i,j} \in C(n, m)$. If $Q = \sum_{i \le -1} b_i V^i$ as predicted by Theorem 19, then bringing $(K-1)Q - P$ to normal form and comparing coefficients of V^i to zero gives the requirement

$$(\tfrac{u}{w})^{(i,0,0)} b'_i - b_i + (\tfrac{v}{w})^{(i-1,0,0)} b'_{i-1} - \sum_{j=0}^{|i|} c_{i,j} \prod_{l=0}^{j-1} (\tfrac{t}{s})^{(i,0,l)} \overset{!}{=} 0.$$

Assuming that we know potential values for b_i we can determine b_{i-1} immediately. Starting from $b_0 = 0$, we can so express all the b_i ($i < 0$) as linear combinations of the undetermined $c_{i,j}$ with coefficients in $C(n, k, m)$. This leads to the following algorithm.

Algorithm 25. Input: $u, v, w, s, t \in C[n, k, m]$
Output: a nonzero operator $P \in C(n, m)\langle V, M\rangle$ such that $(K-1)Q - P \in \mathfrak{a}$ for some $Q \in F\langle V, W, M\rangle$.

1 $P = 0; b_0 = 0;$
2 **for** $i = -1, -2, \ldots$ **do**
3 // in lines 4 and 5, the notation $c_{i,j}$ refers to symbolic variables
4 $b_i = K^{-1} \cdot (\tfrac{w}{v})^{(i,0,0)} \Big(b_{i+1} - (\tfrac{u}{w})^{(i+1,0,0)} b'_{i+1} + \sum_{j=0}^{1-i} c_{i+1,j} \prod_{l=0}^{j-1} (\tfrac{t}{s})^{(i+1,0,l)} \Big)$
5 $P = P + \sum_{j=0}^{1-i} c_{i+1,j} V^{i+1} M^j$
6 **if** $b_i = 0$ for a nontrivial choice of the $c_{l,j} \in C(n, m)$ **then**
7 **return** P with these values in place of the $c_{l,j}$

The condition in line 6 can be checked by clearing denominators of b_i, comparing their coefficients with respect to k to zero and solving the resulting linear system over $C(n, m)$ for the undetermined coefficients $c_{l,j}$. Also note that the undetermined $c_{l,j}$ hidden in the operand (in particular those hidden in $b_{i+1}, b_{i+2}, \ldots$) are not affected by the application of K^{-1} in line 4.

Theorem 26. Algorithm 25 is correct. If f is proper Stirling-like, then the algorithm terminates.

Proof. The output of the algorithm is obviously correct. In order to see that the algorithm terminates when f is proper Stirling-like, observe that Theorem 24 predicts the existence of nontrivial P, Q with $(K-1)Q - P \in \mathfrak{a}$. Then also $(K-1)V^iM^jQ - V^iM^jP \in \mathfrak{a}$ for any $i, j \in \mathbb{Z}$. For appropriate i, j, we have that V^iM^jP only involves terms belonging to $\{\, V^iM^j : i \leq 0, j = 0, \ldots, |i| \,\}$. Therefore, after finitely many iterations, the ansatz for P will cover a set of terms from which an operator can be formed that admits an anti-difference Q. This is when the algorithm terminates. $\square$

Example 27. Consider the sum $g(n,m) = \sum_{k=0}^{m} \binom{m}{k} S_2(k,n)$. Recall from Example 22 that the summand has the annihilator

$$\mathfrak{a} = \left\langle\, (k-m)n + (k-m)N^{-1} + (k+1)K, (1-k+m)M - (m+1) \,\right\rangle,$$

i.e., $V = N^{-1}$, $W = K$, $\alpha = 0$, $\beta = 1$. We make an ansatz $P = \sum_{i \leq 0} \sum_{j=0}^{|i|} c_{i,j} V^i M^j$ ($c_{i,j} \in \mathbb{C}(n,m)$) and construct a corresponding $Q = \sum_{i \leq -1} b_i V^i$ ($b_i \in \mathbb{C}(n,k,m)$).

Coefficient comparison like in Example 22 gives, after eliminating powers of M, first

$$\tfrac{m-k}{k+1} b'_{-1} \stackrel{!}{=} c_{0,0} \quad \Rightarrow \quad b_{-1} = \tfrac{k}{m-k+1} c_{0,0},$$

then in the next step

$$\begin{aligned} &\tfrac{(m-k)(n+1)}{k+1} b'_{-1} - b_{-1} + \tfrac{m-k}{k+1} b'_{-2} \stackrel{!}{=} c_{-1,0} + \tfrac{m+1}{1-k+m} c_{-1,1} \\ \Rightarrow \quad & b_{-2} = \tfrac{k(m+2-k)c_{-1,0} + k(m+1)c_{-1,1} + k(nk+2k-m-mn-2n-3)c_{0,0}}{(k-m-2)(k-m-1)} \end{aligned}$$

By solving a linear system in $\mathbb{C}(n,m)$, it can be found that the choice

$$(c_{0,0}, c_{-1,0}, c_{-1,1}) = (1, n+2, -1)$$

leads to $b_{-2} = 0$. This terminates the algorithm. It follows that

$$(K-1)Q = 1 + (n+2)V^{-1} - V^{-1}M = 1 + (n+2)N - NM$$

for a certain Q. As the sum under consideration has finite support, it follows that the sum $g(n,m)$ satisfies the recurrence

$$g(n+1, m+1) = (n+2)g(n+1, m) + g(n,m).$$

This matches nicely with the defining recurrence for $S_2(m+1, n+1)$. After checking that the sum agrees with $S_2(m+1, n+1)$ for $m = 0$ and all n (for instance), the identity $\sum_{k=0}^{m} \binom{m}{k} S_2(k,n) = S_2(m+1, n+1)$ follows.

In order to find a recurrence of order I with respect to N and M, this algorithm requires roughly $\mathrm{O}(I^7)$ operations in $C(n,m)$. This is because in the Jth iteration there are $\frac{1}{2}J(J+1)$ coefficients $c_{l,j}$ to be determined, which requires roughly $\mathrm{O}(J^6)$ operations. Because the algorithm is applied iteratively for $J = 1, 2, \ldots, I$, this gives a total of $\mathrm{O}(I^7)$.

As for case c), the algorithm is based on coefficient comparison in the representation

$$(K-1)Q \equiv_{\mathfrak{a}} \sum_i \bigl(s_0^{(i-\alpha,0,0)} b'_{i-\alpha} + \cdots + s_\beta^{(i-\alpha-\beta,0,0)} b'_{i-\alpha-\beta} - b_i \bigr) V^i,$$

the s_j emerging from the normal form of $V^\alpha W^\beta$ in the sense of Lemma 11. Also here, the coefficients can be read off one after the other as linear combinations of the undetermined coefficients in P, and as soon as sufficiently many consecutive coefficients b_i are turned to zero by some P, the algorithm stops. For P on the right hand side, it might be necessary to also take up to $|\beta|-1$ powers of W into account in order to guarantee termination (cf. Theorem 24).

Example 28. Consider the sum $g(n,m)=\sum_k \binom{m}{k} S_2(n+2k,k)$. The sequence

$$f\colon \mathbb{Z}^3 \to \mathbb{C}, \quad f(n,k,m)=\binom{m}{k} S_2(n+2k,k)$$

has the annihilator

$$\mathfrak{a}=\big\langle\, (m-k)+(k+1)^2N^{-2}K-(k+1)N^{-1}K, (k-m-1)M+(m+1)\,\big\rangle,$$

i.e., $V=N^{-2}K$, $W=N^{-1}K$, $\alpha=-1$, $\beta=2$. We make an ansatz

$$P=\sum_{i\le 0}\sum_{j=0}^{|i|}\sum_{l=0}^{1} c_{i,j,l} V^i M^j W^l$$

($c_{i,j,l}\in C(n,m)$) and construct a corresponding $Q=\sum_{i\le -1} b_i V^i$ ($b_i \in C(n,k,m)$). We have

$$(K-1)Q \equiv_{\mathfrak{a}} \sum_i \big(((k+1)^2)^{(i-1,0,0)} b'_{i-1} - (\tfrac{(2k+1)(k-m)}{k+1})^{(i,0,0)} b'_i \\ + (\tfrac{(k-m-1)(k-m)}{k(k+1)})^{(i+1,0,0)} b'_{i+1} - b_i\big) V^i.$$

Comparison to P gives

$$\begin{aligned}
b_{-1} &= \tfrac{1}{k} c_{-1,0,1} - \tfrac{m+1}{k(k-m-1)} c_{-1,1,1} + \tfrac{1}{k^2} c_{0,0,0}\\
b_{-2} &= \tfrac{1}{k-1} c_{-2,0,1} - \tfrac{m+1}{(k-1)(k-m-2)} c_{-2,1,1} + \tfrac{(m+1)(m+2)}{(k-1)(k-m-3)(k-m-2)} c_{-2,2,1} + \tfrac{1}{(k-1)^2} c_{-1,0,0}\\
&\quad + \tfrac{k^3-mk^2-2k^2+2mk+3k-m-1}{(k-1)^3k^2} c_{-1,0,1} - \tfrac{m+1}{(k-1)^2(k-m-2)} c_{-1,1,0}\\
&\quad - \tfrac{(m+1)(k^3-mk^2-3k^2+2mk+5k-m-2)}{(k-1)^3k^2(k-m-2)} c_{-1,1,1}\\
&\quad + \tfrac{2k^4-2mk^3-6k^3+5mk^2+9k^2-4mk-5k+m+1}{(k-1)^4k^3} c_{0,0,0}\\
b_{-3} &= \ldots \text{messy} \ldots\\
b_{-4} &= \ldots \text{messy} \ldots
\end{aligned}$$

The choice

$$c_{-2,0,0}=m+1,\ c_{-2,1,0}=-2m-3,\ c_{-2,2,0}=m+2,\\ c_{-3,1,0}=1,\ c_{-3,1,1}=1,\ c_{-3,2,1}=-1$$

(all other $c_{i,j,l}=0$) gives $b_{-2}=b_{-3}=b_{-4}=0$. Thus the algorithm stops and returns $P=V^{-3}M+V^{-3}MW-V^{-3}M^2W+(m+1)V^{-2}-(2m+3)V^{-2}M+(m+2)V^{-2}M^2$.

As a consequence, the sum under consideration satisfies the recurrence equation

$$(m+1)g(n+4,m)-(2m+3)g(n+4,m+1)+(m+2)g(n+4,m+2)\\ +g(n+5,m+1)-g(n+5,m+2)+g(n+6,m+1)=0.$$

6.2. Second Algorithm

We turn back to the situation a). In the second algorithm, we extend the ansatz for P with positive powers of V, i.e., we consider an ansatz of the form

$$P = \sum_{i\geq 0}\sum_{j=0}^{|i|} c_{i,j} V^i M^j$$

with undetermined $c_{i,j}$. Theorem 19 predicts that P will have an anti-difference Q of the form $Q = \sum_{i\geq 0} b_i V^i$, if any. In fact, Theorem 19 does not predict that Q starts with $i = 0$ in the present situation. However, this does not do any harm because P is undetermined: if $i = 0$ turns out to be the wrong choice, this will just cause some of the initial coefficients $c_{i,j}$ to be zero.

Bringing $(K-1)Q - P$ to normal form and comparing coefficients of V^i to zero again gives the requirement

$$(\tfrac{u}{w})^{(i,0,0)} b_i' - b_i + (\tfrac{v}{w})^{(i-1,0,0)} b_{i-1}' - \sum_{j=0}^{|i|} c_{i,j} \prod_{l=0}^{j-1} (\tfrac{t}{s})^{(i,0,l)} \stackrel{!}{=} 0.$$

Assuming that we know potential values for b_{i-1} we can determine b_i by an application of the extended Gosper algorithm. This gives a vector space of potential values for b_i and corresponding values for the $c_{i,j}$. This leads to the following algorithm.

Algorithm 29. Input: $u, v, w, s, t \in C[n, k, m]$
Output: a nonzero operator $P \in C(n, m)\langle V, M\rangle$ such that $(K-1)Q - P \in \mathfrak{a}$ for some $Q \in F\langle V, W, M\rangle$.

1 $b = [\,]; \mathit{PTerms} = [\,];$
2 **for** $i = 0, 1, 2, \ldots$ **do**
3 $\quad \mathit{PTerms} = \mathrm{ListJoin}(\mathit{PTerms}, [V^i, V^i M, \ldots, V^i M^i]);$
4 $\quad b = \mathrm{ListJoin}((-\tfrac{v}{w})^{(i-1,0,0)} K b, \big[1, (\tfrac{t}{s})^{(i,0,0)}, \ldots, \prod_{l=0}^{i-1} (\tfrac{t}{s})^{(i,0,l)}\big]);$
5 $\quad (b, A) = \mathrm{ExtendedGosper}([1, (\tfrac{u}{w})^{(i,0,0)}], b);$
6 $\quad \mathit{PTerms} = A \cdot \mathit{PTerms};$
7 $\quad$ **if** $\exists v \in C(n, m)^{\mathrm{Length}(b)} \setminus \{0\} : v \cdot b = 0$ **then**
8 $\quad\quad$ **return** $v \cdot \mathit{PTerms};$

Theorem 30. Algorithm 29 is correct. If f is proper Stirling-like, then the algorithm terminates.

Proof. Write $Q = \sum_i b_i V^i$. We show the following loop invariant: At the beginning of the ith iteration of the loop (before line 3), the array b contains a basis for the space of potential values of b_{i-1}. For each of these candidates, the PTerms array contains the $C(n, m)$-linear combination of terms $V^j M^l$ that lead to the candidate in the respective component of the b array.

The invariant is clearly true when the loop is entered for the first time, i.e., for $i = 0$. Suppose now it holds at the beginning of the ith iteration. The candidates for the b_i are solutions of a first order inhomogeneous difference equation whose inhomogeneous part is a linear combination of the candidates of the $(i-1)$th iteration, shifted with respect to k and multiplied by $-(\frac{u}{w})^{(i-1,0,0)}$, and the coefficients of the $c_{i,j}$ ($j = 0, \dots, i-1$). This inhomogeneous part is constructed in line 4. Thus, the application of the extended Gosper algorithm in line 5 (Lemma 23 with $C(n,m)$ in place of C) gives a basis of the space of potential values of b_i, and a matrix A containing the coefficients of the linear combinations of the inhomogeneous part which lead to these potential values. After updating the *PTerms* array accordingly (line 6), the loop invariant holds for $i+1$ in place of i.

The loop is terminated as soon as the elements of b are linearly dependent over $C(n,m)$, because then b_i may be set to 0 and all further coefficients of P and Q as well. This completes the construction of $P = v \cdot \mathit{PTerms}$. Note that $v \cdot \mathit{PTerms} \neq 0$, because the rows of the augmented matrix $(A|b)$ as produced by the extended Gosper algorithm are linearly independent.

This completes the proof that the algorithm delivers only correct results. The termination argument is analogous to that in Theorem 26. □

Example 31. Consider once more the sum $g(n,m) = \sum_{k=0}^{m} \binom{m}{k} S_2(k,n)$, and $V, W, \dots$ as in Example 27. We now make an ansatz $P = \sum_{i\geq 0} \sum_{j=0}^{i} c_{i,j} V^i M^j$ and construct a corresponding $Q = \sum_{i \geq 0} b_i V^i$. First,

$$\tfrac{(m-k)n}{k+1} b_0' - b_0 = c_{0,0}$$

has no nontrivial solution. Next, for

$$\tfrac{(m-k)(n-1)}{k+1} b_1' - b_1 = c_{1,0} + c_{1,1} \tfrac{m+1}{1-k+m}$$

the extended Gosper algorithm delivers

$$b = \left(\tfrac{k}{k-1-m}\right) \text{ and } A = \begin{pmatrix} -n & 1 \end{pmatrix}.$$

The operator leading to the nontrivial solution is

$$\bar{V} := A \begin{pmatrix} V \\ VM \end{pmatrix} = -nV + VM.$$

Next, for

$$\tfrac{(m-k)(n-2)}{k+1} b_2' - b_2 = -\bar{c}_1 \tfrac{m-k}{k+1} \left(\tfrac{k}{-1+k-m}\right)' + c_{2,0} + c_{2,1} \tfrac{m+1}{1-k+m} + c_{2,2} \tfrac{m+1}{1-k+m} \tfrac{m+2}{2-k+m}$$

the extended Gosper algorithm delivers

$$b = \left(0, \tfrac{k}{k-m-1}, -\tfrac{k(m+1)}{(k-m-1)(k-m-2)}\right) \text{ and } A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1-n & 0 & 1 & 0 \\ 0 & 0 & 1-n & 1 \end{pmatrix}.$$

The operators leading to these solutions are

$$A \begin{pmatrix} \bar{V} \\ V^2 \\ V^2 M \\ V^2 M^2 \end{pmatrix} = \begin{pmatrix} -\bar{V} + V^2 \\ (1-n)\bar{V} + V^2 M \\ (1-n)V^2 M + V^2 M^2 \end{pmatrix} = \begin{pmatrix} nV - VM + V^2 \\ (n-1)nV - (n-1)VM + V^2 M \\ (1-n)V^2 M + V^2 M^2 \end{pmatrix}.$$

At this point, the entries of b are trivially linearly dependent over $C(n,m)$:

$$\big(0, \tfrac{k}{k-m-1}, -\tfrac{k(m+1)}{(k-m-1)(k-m-2)}\big) \cdot \big(1,0,0\big) = 0.$$

Therefore, we obtain

$$P = \big(1,0,0\big) \cdot \begin{pmatrix} nV - VM + V^2 \\ (n-1)nV - (n-1)VM + V^2 M \\ (1-n)V^2 M + V^2 M^2 \end{pmatrix} = nV - VM + V^2$$

from which the recurrence

$$g(n+1,m+1) = (n+2)g(n+1,m) + g(n,m)$$

follows.

The effect of using the extended Gosper algorithm is that the size of the linear systems to be solved for the termination condition increases slowlier than in the first algorithm, unless the extended Gosper algorithm returns in every iteration a solution space of maximal dimension (which we have never observed in examples). Therefore, the second algorithm is usually superior to the first.

7. Further Examples

We have implemented the algorithms described above in a small Mathematica package, which is available for download at

http://www.risc.uni-linz.ac.at/research/combinat/software/

With the aid of this package, it is an easy matter to prove a lot of identities for sums over Stirling-like terms.

Example 32. Bonus problem 67 of [4] asks for proving that

$$\sum_k (-1)^{m-k} k! \binom{n-k}{m-k} S_2(n+1,k+1) = E_1(n,m) \qquad (n,m \geq 0).$$

Our implementation delivers the recurrence

$$(m-n)g(n,m) - (m+2)g(n,m+1) + g(n+1,m+1) = 0$$

for the sum on the left hand side. The identity is proven by observing that the right hand side satisfies the same recurrence and that the identity holds for $n=0$ and arbitrary m.

Example 33. Bonus problem 68 of [4] is related to the identity

$$\sum_k (-1)^k \binom{2n+1}{k} S_2(n+m+1-k, m+1-k) = E_2(n,m) \qquad (n,m \geq 0).$$

Our implementation delivers the recurrence

$$(2n-m)g(n,m) + (m+2)g(n,m+1) - g(n+1,m+1) = 0$$

for the sum on the left hand side. The identity is proven by observing that the right hand side satisfies the same recurrence and that the identity holds for $n=0$ and arbitrary m.

Our algorithm can be easily modified such as to cover also differential operators. This is useful for proving identities about Bernoulli polynomials, for instance.

Example 34. Consider the identity

$$\sum_k \binom{m}{k} y^{m-k} B_k(x) = B_m(x+y),$$

where $B_n(x)$ denotes the nth Bernoulli polynomial [11]. These polynomials satisfy

$$D_x B_n(x) - n B_{n-1}(x) = 0.$$

A slightly modified version of our algorithm finds that the sum satisfies the differential equation

$$D_x f(m+1, x, y) - (m+1) f(m, x, y) = 0.$$

The identity is proven by observing that the right hand side satisfies the same recurrence and that the identity holds for $m = 0$ and arbitrary x and y, and for $x = y = 0$ and arbitrary m.

8. Concluding Remarks

We have shown that sums over proper Stirling-like terms with at least two free variables satisfy a linear multivariate (partial) recurrence equation with polynomial coefficients, and that such a recurrence can be efficiently computed given the annihilator of the Stirling-like summand. Many summation identities about Stirling-numbers that can be found in the literature can be verified by this algorithm. Whether our algorithm is acceptable as a solution to the research problem posed by Graham, Knuth, and Patashnik mainly depends on the interpretation of the phrase "terms that may involve Stirling numbers" they use.

It might be desirable to have an extension of the algorithm that would allow the product of two or more Stirling-like terms to arise in the summand. We expect that Theorem 14 generalizes to such terms, provided that the number of free variables is increased further such as to exceed the ideal dimension of the annihilator.

Our algorithms can be applied also if no additional free variables (besides n) are present. In this situation they will terminate if and only if the summand of the sum under consideration admits a k-free recurrence. This way, for instance the conversion formulas

$$\sum_{k=0}^{n} S_1(n,k) x^k = x^{\underline{n}} \quad \text{and} \quad \sum_{k=0}^{n} S_2(n,k) x^{\underline{k}} = x^n$$

can be found automatically. An algorithm for deciding the existence of such a recurrence would be interesting both for theoretical and for practical reasons.

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