An Algorithm for Deciding Zero Equivalence of Nested Polynomially Recurrent Sequences

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We introduce the class of nested polynomially recurrent sequences which includes a large number of sequences that are of combinatorial interest. We present an algorithm for deciding zero equivalence of these sequences, thereby providing a new algorithm for proving identities among combinatorial sequences: in order to prove an identity, decide by the algorithm whether the difference of left hand side and right hand side is identically zero. This algorithm is able to treat mathematical objects which are not covered by any other known symbolic method for proving combinatorial identities. Despite its theoretical flavor and its high complexity, an implementation of the algorithm can be successfully applied to nontrivial examples.

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1. INTRODUCTION

Computer proofs of special function identities came up in the early nineties when Zeilberger [Zeilberger 1990] presented algorithms for deciding whether a representation of a *holonomic function* represents the zero function. Holonomic functions are solutions of systems of differential-difference equations of a certain shape. The study of holonomic (and so-called ∂-finite) functions [Chyzak and Salvy 1998; Chyzak 2000] was motivated by the observation that many special functions are of such type and that the defining differential-difference system of such functions provides a convenient representation of the mathematical object which can be used in computations.

In this paper, we restrict our attention to univariate sequences, i.e., functions with domain $\mathbb{N}$. In this case, a sequence $(f(n))_{n\geq 0}$ is called holonomic if it satisfies a linear homogeneous recurrence relation with polynomial coefficients, i.e., we have

$$p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r) = 0 \quad (n \geq 0)$$

for certain polynomials $p_1, \ldots, p_r$. Univariate holonomic sequences are also called

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P-finite [Stanley 1999] and there are computer algebra packages available for dealing with such sequences [Salvy and Zimmermann 1994; Mallinger 1996]. However, little is known about the algorithmic treatment of functions which are not holonomic. We present an algorithm for proving zero equivalence (and hence, for proving identities) of *nested polynomially recurrent sequences,* to be defined in Section 3. The class of these sequences contains all holonomic sequences, but it contains in addition also plenty of interesting objects which are not holonomic and which, to our knowledge, could not be handled so far by symbolic methods. Section 6 contains some examples. A similar algorithm for deciding zero equivalence for a certain class of analytic functions was given by Shackell [Shackell 1993].

We will employ the notions of *difference algebra* [Cohn 1965], which can be seen as a discrete analogue to differential algebra [Ritt 1950]. Sequences will be defined by annihilating *difference polynomials* from a polynomial *difference ring* whose variables represent the sequences under consideration, or shifts of these sequences (see Section 2). Compared to the definition of sequences by annihilating linear operators, as it is used in algorithms for holonomic objects [Zeilberger 1990; Chyzak and Salvy 1998; Chyzak 2000], the use of difference polynomials allows the definition of sequences of a more general type, as will be shown in Section 3. However, we employ the notion of difference algebra only as a convenient language, but we will not need any deep results from the theory of difference algebra. The basic definitions we need will be stated in Section 2. For our arguments we assume familiarity with elementary concepts of computational commutative algebra only, as it is presented, e.g., in [Cox et al. 1992].

On one hand, our algorithm is of theoretical interest. It provides a decision procedure for deciding zero equivalence of nested polynomially recurrent sequences. No such decision procedure was known before. On the other hand, our algorithm is of practical relevance, for instance, to prove entries from mathematical tables like [Abramowitz and Stegun 1972] (see also Example 6.4). Despite its very high worst case complexity, we succeeded in proving nontrivial identities using an implementation of the algorithm in frame of a Mathematica package [Kauers 2006].

This paper is organized as follows. Section 2 introduces some basic notions and convenient notation. In Section 3 we introduce the class of nested polynomially recurrent sequence. We will give some examples and some closure properties of the class of these sequences. Section 4 presents how nested polynomially recurrent sequences are translated into the language of difference rings by means of *defining relations.* Section 5 presents the algorithm for proving zero equivalence, along with proofs of its correctness and termination. Section 6 has a collection of examples that can be tackled by the algorithm. The paper is concluded by some remarks about the efficiency of the algorithm in Section 7.

2. DIFFERENCE RINGS AND DIFFERENCE IDEALS

Let K be a computable field of characteristic zero. By computable, we mean that every element $a \in K$ should have a finite representation $\bar{a}$ and for any representation $\bar{a}$ of an element in $a \in K$ it should be decidable if a is the zero element of K, and for any two representations $\bar{a}, \bar{b}$ of field elements $a, b \in K$, representations of $a + b$ and $a \cdot b$ should be computable. Natural choices for K are number fields of finite degree or finite transcendental extensions of $\mathbb{Q}$.

We recall the basic definitions of difference algebra [Cohn 1965]. A *difference ring* R is a commutative ring, equipped with an endomorphism $s\colon R \to R$. The endomorphism s is called the *shift operator* of R. An element $r \in R$ is called *constant* if $s(r) = r$.

An important example of a difference ring is the m-fold *polynomial difference ring,* to be denoted by $R^{(m)}$. Let $\{t_{i,j} : i = 1, \dots, m, j \in \mathbb{N}_0\}$ be algebraically independent over K and let s be canonically defined by $s(c) = c$ $(c \in K)$ and $s(t_{i,j}) := t_{i,j+1}$. Then

$$R^{(m)} := K[t_{1,0}, \dots, t_{m,0}, t_{2,1}, \dots, t_{m,1}, t_{3,1}, \dots\dots],$$

equipped with this s, forms a difference ring. We view $R^{(m)}$ as a polynomial ring with infinitely many indeterminates. The elements of $R^{(m)}$ are called *difference polynomials.* Readers familiar with differential algebra will note the similarity with differential polynomials. However, it is worth noting that $s(ab) = s(a)s(b)$ whereas the derivative D in a differential ring obeys the more complicated Leibniz rule $D(ab) = D(a)b + aD(b)$.

Writing $s^n := s \circ s^{n-1}$, $s^0 = \mathrm{id}$, defining $t_i := t_{i,0}$ $(i = 1, \dots, m)$ and omitting parentheses we will often write $s^j t_i$ in place of $t_{i,j}$. This shall remind us that the index j corresponds to the jth shift of the object represented by the variable t_i. We will use similar shortcuts not only for the indeterminates, but also for polynomials, sets of polynomials, etc. As an example, $sp = s^3t_2 + s^2t_3 + st_1$ if $p = s^2t_2 + st_3 + t_1$.

Though we understand $R^{(m)}$ as a polynomial ring with infinitely many indeterminates, each particular difference polynomial $p \in R^{(m)}$ only involves finitely many of them. We introduce *restrictions* $R_r^{(m)}$ of $R^{(m)}$ where only shifts up to some finite order $r \in \mathbb{N}$ appear. We define

$$R_r^{(m)} := K[t_{1,0}, t_{2,0}, \dots, t_{m,0}, \dots\dots t_{1,r}, \dots, t_{m,r}] \subseteq R^{(m)} \qquad (r \in \mathbb{N}).$$

This is a polynomial ring over K in $m(r+1)$ indeterminates. The shift operator s on $R^{(m)}$ does not induce a shift operator of $R_r^{(m)}$ because $s(t_{i,r}) \notin R_r^{(m)}$ $(i = 1, \dots, m)$. We call r the *order* and m the *depth* of $R_r^{(m)}$. As m will be fixed in all our considerations, we can safely write $R := R^{(m)}$ and $R_r := R_r^{(m)}$ for short. Observe that every difference polynomial $p \in R$ also belongs to R_r provided that r is large enough.

A *difference ideal* I in some difference ring R is an ideal in R such that $sI \subseteq I$. If $S \subseteq R$ is any subset of R and I is the intersection of all difference ideals in R containing S, then we say I is *generated* by S and S is called a *basis* of I. We write $I = \langle\langle S \rangle\rangle$. Note that if S itself has the property that $sS \subseteq S$ then I is the usual ring ideal generated by S, denoted by $I = \langle S \rangle$.

For $S \subseteq R_r$, we write $\langle S \rangle_r$ for the polynomial ideal generated by S in R_r. Note that $\langle S \rangle_r \neq \langle S \rangle_{r+1}$ unless $S = \{0\}$. The notation $I \trianglelefteq R_r$ expresses that the set $I \subseteq R_r$ is a polynomial ideal in R_r, i.e., $I = \langle I \rangle_r$. If $I \trianglelefteq R_r$ is some ideal, then $\mathrm{Rad}\, I$ denotes the radical of I.

3. NESTED POLYNOMIALLY RECURRENT SEQUENCES

We have already fixed a computable field K of characteristic zero in the previous section. Let the class $\mathcal{C}$ of sequences $\mathbb{N} \to K$ be defined by structural induction as follows. Let $f_1, \ldots, f_s \in \mathcal{C}$ and $r \geq 0$. A sequence $f = (f(n))_{n=1}^{\infty}$ in K belongs to $\mathcal{C}$ if it satisfies a recurrence relation of the form

$$\begin{aligned} f(n+r) = p\big(\; & f_1(n),\ f_1(n+1),\ \ldots\ldots,\ f_1(n+r-1),\ f_1(n+r), \\ & f_2(n),\quad \ldots\ldots \quad \ldots\ldots,\ f_2(n+r-1),\ f_2(n+r), \\ & \quad\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ & f_s(n),\quad \ldots\ldots \quad \ldots\ldots,\ f_s(n+r-1),\ f_s(n+r), \\ & f(n),\ f(n+1),\ \ldots\ldots,\ f(n+r-1)\;\big) \end{aligned}$$

where p may be (1) a polynomial function or (2) the reciprocal of a polynomial function. (In other words: p is a rational function with constant numerator or constant denominator.) The induction base is given by the case $s = 0$.

We call $\mathcal{C}$ the class of *nested polynomially recurrent sequences.* Its elements are "polynomially recurrent" in the sense that p is not limited to polynomials that are linear in $f(n+i)$. Nonlinear recurrences are allowed as well. The term "nested" reflects the fact that the definition of f may involve other nested polynomially recurrent sequences.

The number r in the definition is called the *order* of f. The sequences $f_1, \ldots, f_s$ from the definition are called *subexpressions* of f, and the notion of subexpression is understood transitively. The total number m of subexpressions of f is called the *depth* of f. The notions of depth and order depend on the definition of a sequence rather than on the sequence itself, e.g., $(1/n - 1/n)_{n=1}^{\infty}$ has order 0 and depth 2, $(f(n))_{n=1}^{\infty}$ with $f(n+2) := 3f(n)$, $f(1) = f(2) = 0$ has order 2 and depth 1, and $(0)_{n=1}^{\infty}$ has depth 0 and order 0, yet all three sequences are equal, only their representations differ.

The class $\mathcal{C}$ contains a large variety of sequences which appear frequently in practice. It is immediate that all holonomic sequences are contained in $\mathcal{C}$. In addition, $\mathcal{C}$ contains sequences like $(\alpha^{\beta^n})_{n=1}^{\infty}$ ($\alpha \in K, \beta \in \mathbb{Z}$; by $f(n+1) = f(n)^{\beta}$) or $(\alpha^{\mathrm{F}(n)})_{n=1}^{\infty}$ ($\alpha \in K$, $\mathrm{F}(n)$ the nth Fibonacci number; by $f(n+2) = f(n+1)f(n)$) which are easily seen not to be holonomic. If $k \in \mathbb{N}$ is fixed, then also the sequences $(s(n,k))_{n\geq 0}$, $(S(n,k))_{n\geq 0}$, $(e(n,k))_{n\geq 0}$, and $(E(n,k))_{n\geq 0}$ of Stirling and Eulerian numbers of first and second kind [Graham et al. 1994] belong to $\mathcal{C}$. A class of nonlinear recurrent sequences which arise in combinatorial and number theoretic considerations is studied in [Aho and Sloane 1973; Golomb 1963]. These sequences satisfy recurrences of the form $f(n+1) = f(n)^2 + \alpha f(n) + \beta$ for certain $\alpha, \beta \in \mathbb{Q}$, thus they are members of $\mathcal{C}$. An example is Sylvester's sequence [Sloane and Plouffe 1995, M0865]. Solutions of the "quadratic" Fibonacci recurrence [Duke et al. 1998] $h(n+2) = h(n+1) + h(n)^2$ also belong to $\mathcal{C}$.

The Handbook of Mathematical Functions [Abramowitz and Stegun 1972] contains a lot of families $f_n(x)$ of special functions which, for fixed x, admit sequences in n which belong to $\mathcal{C}$.

EXAMPLE 3.1. *The following quantities may be regarded as elements of $\mathcal{C}$:*

(1) *The exponential integral [Abramowitz and Stegun 1972, p. 227ff.]*

$$E_n(x) := \int_1^\infty t^{-n} \exp(-xt)\, dt$$

by $E_{n+1}(x) = (\exp(-x) - xE_n(x))/n$,

(2) *The incomplete Gamma function [Abramowitz and Stegun 1972, p. 260ff.]*

$$\Gamma(n,x) := \int_x^\infty t^{n-1} \exp(-t)\, dt$$

by $\Gamma(n,x) = \Gamma(n-1,x)(n-1) + x^{n-1}\exp(-x)$,

(3) *the quantile of the* χ^2*-distribution [Abramowitz and Stegun 1972, p. 940ff.]*

$$Q(\chi^2|n) := 1 - \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^{\chi^2} t^{n/2-1} \exp(-t/2)\, dt$$

by $Q(\chi^2|n+2) = Q(\chi^2|n) + (\chi^2/2)^{n/2}\exp(-\chi^2/2)/\Gamma(n/2+1)$.

If these functions can be handled for every fixed $n \in \mathbb{N}$*, then our algorithm provides a tool to prove relations among them for general* n*.*

It is clear that $\mathcal{C}$ is closed under field operations provided that they are meaningful, i.e., denominators must not vanish anywhere on the natural numbers. It is also quite clear that $\mathcal{C}$ is closed under taking indefinite sums and products, for $F(n) = \sum_{k=1}^n f(k)$ satisfies $F(n+1) = F(n) + f(n+1)$ and $F(n) = \prod_{k=1}^n f(k)$ satisfies $F(n+1) = f(n+1)F(n)$. It follows that $\mathcal{C}$ contains all $\Pi\Sigma$-sequences [Karr 1981], i.e., all sequences that can be represented by expressions involving rational functions and indefinite summation and product signs. Many definite sums $F(n) = \sum_{k=1}^n f(k,n)$ obey linear recurrences and therefore also belong to $\mathcal{C}$. For large classes of summands $f(k,n)$, suitable recurrences for $F(n)$ can be computed by the methods of Chyzak [Chyzak 2000], Schneider [Schneider 2001] or Zimmermann [Zimmermann pear], and a recurrence computed by one of these methods can be used as a definition of F in the present context.

It may be remarked that $\mathcal{C}$ is also closed under taking indefinite continued fractions. Given $(f(n))_{n=1}^\infty \in \mathcal{C}$ with $f(n) \neq 0$ $(n \in \mathbb{N})$ we introduce the notation

$$F(n) := \mathop{\mathrm{K}}_{k=1}^{n} f(k) := f(1) + {}^{1}\big/ f(2) + {}^{1}\big/ \cdots + {}^{1}\big/ f(n).$$

It is at the heart of the theory of continued fractions [Perron 1929] that $F(n)$ can be written as a quotient $\varkappa_1(n)/\varkappa_2(n)$ where

$$\varkappa_1(n+2) = f(n+2)\varkappa_1(n+1) + \varkappa_1(n),\ \varkappa_1(1) = f(1),\ \varkappa_1(2) = 1 + f(1)f(2),$$
$$\varkappa_2(n+2) = f(n+2)\varkappa_2(n+1) + \varkappa_2(n),\ \varkappa_2(1) = 1,\ \varkappa_2(2) = f(2).$$

Obviously, $(\varkappa_1(n))_{n=1}^\infty, (\varkappa_2(n))_{n=1}^\infty \in \mathcal{C}$, and as $\mathcal{C}$ is closed under arithmetic operations, it follows $(F(n))_{n=1}^\infty \in \mathcal{C}$.

4. DEFINING RELATIONS

Let $f \in \mathcal{C}$ be given. The goal of this section is the construction of a finite set $D \subseteq R_r$ of *defining relations* for f, according to the following definition.

Definition 4.1. Let $f_1, \dots, f_m \in \mathcal{C}$, $r \in \mathbb{N}$. A finite set $D \subseteq R_r$ is called a *set of defining relations* for $f_1, \dots, f_m$ if

(1) For all $n \in \mathbb{N}$, the ideal

$$\left\langle D \cup \{s^i t_j - f_j(n+i) : i = 0, \dots, r-1, j = 1, \dots, m\}\right\rangle_r$$

has a unique point, and in this solution $s^r t_j = f_j(n+r)$ $(j = 1, \dots, m)$.

(2) For every $j \in \{1, \dots, m\}$, there exists exactly one polynomial $p \in D$ of the form $p = s^r t_j + q$ or $p = q s^r t_j - 1$ where q depends only on $s^{r'} t_{j'}$ with $j' \leq j$ and $r' \leq r$, but not on $s^r t_j$. This p is called the *defining polynomial* or the *defining relation* of $s^r t_j$.

(3) For all $p \in R_{r-1}$, we have $p \in D \iff sp \in D$.

The number r is called the *order* of D. We say that the sequence f_i *corresponds* to the variable t_i and write $t_i \sim f_i$. If $f_1, \dots, f_{m-1}$ are the subexpressions of f_m, then we also say D is a set of defining relations for f_m.

Defining relations define the sequence up to initial values: if a set D of defining relations for $f_1, \dots, f_m \in \mathbb{C}$ is given, and D is of order r, then all values $f_i(n)$ can be computed as soon as initial values $f_i(j)$ $(i = 1, \dots, m; j = 1, \dots, r)$ are fixed.

We next collect some important properties of sets of defining relations. If we say that the set $A \cup sA \subseteq R_{r+1}$ is obtained on shift of the set $A \subseteq R_r$, then the essence of the following lemma is that the property of being a set of defining relations is preserved under shift.

LEMMA 4.2. *Let $f \in \mathcal{C}$ be a nested polynomially recurrent sequence of depth m and $D \subseteq R_r$ be a set of defining relations for f. Let $f_1, \dots, f_m$ be the sequences corresponding to the variables $t_1, \dots, t_m$, respectively. Then $D \cup sD \subseteq R_{r+1}$ is also a set of defining relations for f.*

PROOF. It is clear that the conditions (2) and (3) of Def. 4.1 are satisfied for $D \cup sD$. As for (1), take an arbitrary $n \in \mathbb{N}$. Then,

$$\left\langle D \cup \{s^i t_j - f_j(n+i) : i = 0, \dots, r-1,\ j = 1, \dots, m\}\right\rangle_r$$

has a unique solution with $s^r t_j = f_j(n+r)$ $(j = 1, \dots, m)$ and

$$\left\langle sD \cup \{s^i t_j - f_j(n+i) : i = 0, \dots, r,\ j = 1, \dots, m\}\right\rangle_{r+1}$$

has a unique solution with $s^{r+1} t_j = f_j(n+r+1)$ $(j = 1, \dots, m)$. It follows that

$$\left\langle (D \cup sD) \cup \{s^i t_j - f_j(n+i) : i = 0, \dots, r,\ j = 1, \dots, m\}\right\rangle_{r+1}$$

also as a unique solution with $s^{r+1} t_j = f_j(n+r+1)$ $(j = 1, \dots, m)$. $\square$

The following proposition states that the polynomial ideal generated by a set of defining relations in some R_r coincides with the intersection of R_r with the difference ideal it generates in R. The proof of the proposition proceeds by considering lexicographic Gröbner bases and using the elimination property in polynomial rings with finitely many variables. As the proposition is not needed in the sequel, we omit the details of the proof.

Proposition 4.3. *If D is a set of defining relations of order r and $\langle\langle D\rangle\rangle$ is the difference ideal generated by D in R, then $\langle\langle D\rangle\rangle \cap R_r = \langle D\rangle_r$.*

We now turn to the construction of sets of defining relations for the elements of $\mathcal{C}$. Given $f \in \mathcal{C}$ with subexpressions $f_1, \ldots, f_{m-1}$, a set of defining relations can easily be obtained using the recurrences fulfilled by the f_l ($l = 1, \ldots, m-1$) and $f_m := f$. Suppose the f_l are numbered such that all subexpressions of f_l are among the $f_1, \ldots, f_{l-1}$, let r_l be the order of f_l and put $r := \max_l r_l$. We distinguish two cases, according to the two cases in the definition of $\mathcal{C}$.

(1) $f_l(n + r_l) = p\big(f_1(n), \ldots, f_i(n + j), \ldots, f_l(n + r_l - 1)\big)$ for some polynomial function p. For such f_l, put $d_l := s^{r_l} t_l - p(t_1, \ldots, s^j t_i, \ldots, s^{r_l-1} t_l) \in R_{r_l}$.

(2) $f_l(n + r_l) = 1/p\big(f_1(n), \ldots, f_i(n + j), \ldots, f_l(n + r_l - 1)\big)$ for some polynomial function p. For such f_l, put $d_l := p(t_1, \ldots, s^j t_i, \ldots, s^{r_l-1} t_l) s^{r_l} t_l - 1 \in R_{r_l}$.

Using this notation, define $D \subseteq R_r$ as

$$D := \{d_1, sd_1, \ldots, s^{r-r_1} d_1,\ d_2, sd_2, \ldots, s^{r-r_2} d_2,\ \ldots\ldots,\ d_m, sd_m, \ldots, s^{r-r_m} d_m\}.$$

It is immediate by construction that D is a set of defining relations for f. In practice, we will of course represent common subexpressions by a single variable queue $t_l, st_l, sst_l, \ldots$ rather than by separate ones, and more subtle optimizations for reducing the number of variables are thinkable as well.

Example 4.4. *Consider the sequence $(f(n))_{n=1}^{\infty} \in \mathcal{C}$ defined by*

$$f(n) := \frac{\mathrm{F}(n)}{\mathrm{F}(n+1)} + \sum_{k=1}^{n} \frac{(-1)^k}{\mathrm{F}(k)\,\mathrm{F}(k+1)},$$

where $\mathrm{F}(n)$ denotes the nth Fibonacci number. An appropriate set of defining relations for f is

$$\begin{aligned}
D = \{&st_1 + t_1, sst_1 + st_1, && (\,t_1 \sim (-1)^n\,)\\
&sst_2 - st_2 - t_2, && (\,t_2 \sim \mathrm{F}(n)\,)\\
&sst_3 - st_3 - t_3, && (\,t_3 \sim \mathrm{F}(n+1)\,)\\
&t_4 t_3 - 1, st_4 st_3 - 1, sst_4 sst_3 - 1, && (\,t_4 \sim 1/\,\mathrm{F}(n+1)\,)\\
&t_5 t_2 t_3 - 1, st_5 st_2 st_3 - 1, sst_5 sst_2 sst_3 - 1, && (\,t_5 \sim 1/\,\mathrm{F}(n)\,\mathrm{F}(n+1)\,)\\
&st_6 - t_6 - st_1 st_5, sst_6 - st_6 - sst_1 sst_5, && (\,t_6 \sim \Sigma_{k=1}^{n} \ldots\,)\\
&t_7 - t_2 t_4 - t_6, st_7 - st_2 st_4 - st_6, && (\,t_7 \sim f(n)\,)\\
&\qquad sst_7 - sst_2 sst_4 - sst_6\,\}
\end{aligned}$$

This representation was obtained by mechanically applying the definitions of the various subexpressions, and representing identical subexpressions by the same variable. However, a "better" set of defining relations for f can be obtained by exploiting *that* $t_2 \sim \mathrm{F}(n)$ *implies* $st_2 \sim \mathrm{F}(n+1)$.

$$
\begin{aligned}
D' = \{&t_1 + st_1, st_1 + sst_1, && (\, t_1 \sim (-1)^n \,)\\
&sst_2 - st_2 - t_2, && (\, t_2 \sim \mathrm{F}(n)\,)\\
&t_3t_2 - 1, st_3st_2 - 1, sst_3sst_2 - 1, && (\, t_3 \sim 1/\,\mathrm{F}(n)\,)\\
&st_4 - t_4 - t_1t_2st_3, sst_4 - st_4 - st_1st_2sst_3, && (\, t_4 \sim \Sigma_{k=1}^{n-1} \dots)\\
&st_5 - t_2st_3 - st_4, sst_5 - st_2sst_3 - sst_4 \,\} && (\, t_5 \sim f(n-1)\,).
\end{aligned}
$$

Given an expression in standard mathematical notation as in the example above, it is not hard to construct a set of defining relations. The SumCracker package [Kauers 2006], implementing the algorithm of this paper, is able to do this task automatically for a fairly large set of expressions.

5. PROVING ZERO EQUIVALENCE

We now turn to the algorithm for deciding $f \stackrel{?}{=} 0$ for elements $f \in \mathcal{C}$ given by a set D of defining relations and initial values.

The key idea is an induction argument. The algorithm computes a number $k \in \mathbb{N}$ such that $f(n) = \dots = f(n+k-1) = 0$ implies $f(n+k) = 0$ for arbitrary $n \in \mathbb{N}$. After that, f is evaluated at k consecutive points, and either there is a counterexample among these values, or there is no counterexample at all.

Algorithm 5.1.

Input:	f — *a nested polynomially recurrent sequence*
	D — *a set of defining relations for* f
Output:	true *or* false*, depending on whether* $f \equiv 0$ *or not*
Assumptions:	t_m *corresponds to* $(f(n))_{n=1}^{\infty}$*,* D *is of order* r

1 **function** isZeroEquivalent(f, D)
2   $k \leftarrow 0$
3   $I_0 \leftarrow \langle D\rangle_r + \langle t_m, st_m, \dots, s^{r-1}t_m\rangle_r$
4   **while** $s^{k+r}t_m \notin \operatorname{Rad} I_k$ **do**
5     $k \leftarrow k+1$
6     $I_k \leftarrow \langle I_{k-1}\rangle_{k+r} + \langle s^{r+k-1}t_m\rangle_{k+r} + \langle s^k D\rangle_{k+r}$
7   **end do**
8   **for** n **from** 1 **to** $k+r$ **do**
9   **if** $f(n) \neq 0$ **then**
10   **return** false, counterexample $= n$
11   **return** true

Note that all steps of the algorithm are computable: a basis for the sum of two ideals is obtained by concatenating the bases of the summands, and the radical membership test in line 4 can be decided by means of Gröbner bases and the Rabinowitsch trick [Cox et al. 1992]. Hilbert's Nullstellensatz implies that $f \in \operatorname{Rad}\langle f_1, \dots, f_n\rangle$ if and only if vanishing of all the f_i implies vanishing of f. This is the core of the correctness argument which is made explicit next.

Theorem 5.2. *Algorithm 5.1 is correct.*

Proof. It is clear that $f \not\equiv 0$ whenever the algorithm returns "false" because this does only happen when a counterexample has been found. Now suppose the algorithm returns "true.". We will prove $f(n) = 0$ ($n \in \mathbb{N}$) by induction on n.

First, according to lines 8–10, we have $f(1) = \cdots = f(k + r) = 0$ as base of the induction. Now let $n \in \mathbb{N}$ be arbitrary such that $f(n) = f(n + 1) = \cdots = f(n + k + r - 1) = 0$. Prove $f(n + k + r) = 0$.

By repeated application of Lemma 4.2, $D \cup sD \cup \cdots \cup s^k D$ is a set of defining relations for f because D is. Let f_j ($j = 1, \ldots, m$) be the sequences corresponding to the variables t_j ($j = 1, \ldots, m$), respectively. By assumption of the algorithm, $f_m = f$. Condition (1) of Def. 4.1 asserts that the ideal

$$J = \left\langle D \cup \cdots \cup s^k D \cup \{s^i t_j - f_j(n + i) : i = 0, \ldots, k + r - 1, j = 1, \ldots, m\} \right\rangle_{k+r}$$

has a unique point, and this point has the coordinates $s^{k+r} t_j = f_j(n + k + r)$. Now, by induction hypothesis, $I_k \subseteq J$. Every solution of J must be a solution of I_k as well. But by the termination condition in line 4, I_k has only solutions with $s^{k+r} t_m = 0$. This implies $f(n + k + r) = 0$. □

The next theorem will assert the termination of Algorithm 5.1. For its proof, we will need two technical lemmas.

Lemma 5.3. *Let $\mathfrak{p} \trianglelefteq K[X] =: K[x_1, \ldots, x_n]$ be a prime ideal. By $Q(R)$, we denote the quotient field of an integral domain R. Then*

(1) *For all $q \in K[X]$, the ideal $\mathfrak{p}' := \langle \mathfrak{p} \cup \{p\} \rangle \trianglelefteq K[X, y]$ with $p = y - q$ is prime and the quotient fields of the coordinate rings are isomorphic, $Q(K[X]/\mathfrak{p}) \cong Q(K[X, y]/\mathfrak{p}')$.*

(2) *For all $q \in K[X] \setminus \mathfrak{p}$, the ideal $\mathfrak{p}' := \langle \mathfrak{p} \cup \{p\} \rangle \trianglelefteq K[X, y]$ with $p = qy - 1$ is prime and $Q(K[X]/\mathfrak{p}) \cong Q(K[X, y]/\mathfrak{p}')$.*

Proof. Let $R = K[X]/\mathfrak{p}$ and $R' = K[X, y]/\mathfrak{p}'$.

(1) $p = y - q$ for $q \in K[X]$. Consider the homomorphisms ϕ, ψ defined by

$$\begin{aligned} &\phi\colon K[X] \to R' && x_i \mapsto x_i \ (i = 1, \ldots, n), \\ &\psi\colon K[X, y] \to R && x_i \mapsto x_i \ (i = 1, \ldots, n), \quad y \mapsto q(x_1, \ldots, x_n). \end{aligned}$$

As $\mathfrak{p} \subseteq \ker \phi$ and $\mathfrak{p}' \subseteq \ker \psi$, these homomorphisms induce homomorphisms $\bar{\phi}\colon R \to R'$ and $\bar{\psi}\colon R' \to R$. $\bar{\phi}$ and $\bar{\psi}$ are inverses of each other because $\bar{\psi}(\bar{\phi}(x_i)) = \bar{\phi}(\bar{\psi}(x_i)) = x_i$ for $i = 1, \ldots, n$ and $\bar{\phi}(\bar{\psi}(y)) = \bar{\phi}(q) = q \equiv_{\mathfrak{p}'} y$. It follows that $R \cong R'$ and consequently $Q(R) \cong Q(R')$.

(2) $p = qy - 1$ for $q \in K[X] \setminus \mathfrak{p}$. This case is not as immediate as case (1) because R and R' themselves need not be isomorphic. But we still have $Q(R) \cong Q(R')$: By $\mathfrak{p}' = \langle \mathfrak{p} \rangle + \langle p \rangle$ we have $R' \cong R[y]/\langle p \rangle$. It suffices to show the existence of an embedding $R' \hookrightarrow Q(R)$, for then $R \hookrightarrow R' \hookrightarrow Q(R)$, and hence $Q(R) \hookrightarrow Q(R') \hookrightarrow Q(R)$, and hence $Q(R') \cong Q(R)$.

As $q \notin \mathfrak{p}$, there is some element $1/q \in Q(R)$. The evaluation homomorphism $\phi\colon R[y] \to Q(R)$, $\phi(y) = 1/q$ induces a homomorphism $\bar{\phi}\colon R' \to Q(R)$ because $\langle p \rangle \subseteq \ker \phi$. If furthermore $\ker \phi \subseteq \langle p \rangle$ then $\bar{\phi}$ is injective, and we are done. Indeed, $\ker \phi \subseteq \langle p \rangle$: Let $a = \sum_{i=0}^{n} a_i y^i \notin \langle qy - 1 \rangle$ be in canonical form, i.e., fully reduced wrt. $qy-1$. Then $q \nmid a_n$ in $K[X]$. Suppose $a \in \ker\phi$, i.e.,

$$0 = \phi(a) = \sum_{i=0}^{n} a_i \phi(y)^i = \sum_{i=0}^{n} \frac{a_i}{q^i} = \frac{1}{q^n} \sum_{i=0}^{n} a_i q^{n-i}.$$

As $1/q^n \neq 0$, it follows that

$$0 = \sum_{i=0}^{n} a_i q^{n-i} = a_n + q \underbrace{\sum_{i=0}^{n-1} a_i q^{(n-1)-i}}_{\in K[X]},$$

and hence $q \mid a_n$, a contradiction. □

For the present context, the previous lemma provides an invariant property for certain ideals in R_k upon extending them to ideals in R_{k+1}. This observation is crucial for the termination proof. Recall [Cox et al. 1992] that the dimension of a prime ideal $\mathfrak{a} \trianglelefteq K[X]$ may be defined as the transcendence degree of $Q(K[X]/\mathfrak{a})$ over K. Every ideal $\mathfrak{a}$ can be written as a finite intersection $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$ where the $\mathrm{Rad}(\mathfrak{p}_i)$ are uniquely determined pairwise distinct prime ideals, called the associated prime ideals of $\mathfrak{a}$. The dimension of an arbitrary ideal $\mathfrak{a}$ is then defined as the maximum of the dimensions of the associated prime ideals of $\mathfrak{a}$.

Lemma 5.4. *Let $D \subseteq R_r$ be a set of defining relations for some $f \in \mathcal{C}$, where r is the order of D. For all $k \geq 0$, define $J_k := \langle D \cup sD \cup \cdots \cup s^k D \rangle_{k+r}$. Furthermore, let $J_k \subseteq \mathfrak{a} \trianglelefteq R_{k+r}$ and $\mathfrak{a}' := \langle \mathfrak{a} \rangle_{k+r+1} + J_{k+1} \trianglelefteq R_{k+r+1}$ for some fixed $k \in \mathbb{N}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \trianglelefteq R_{k+r}$ be the associated prime ideals of $\mathfrak{a}$. Then*

(1) *The ideals $\mathfrak{p}_i + J_{k+1}$ $(i = 1, \ldots, s)$ are the associated primes of $\mathfrak{a}'$.*

(2) *The dimension of $\mathfrak{p}_i$ in R_{k+r} is equal to the dimension of $\mathfrak{p}_i + J_{k+1}$ in R_{k+r+1} $(i = 1, \ldots, s)$.*

Proof. As $J_k \subseteq \mathfrak{a}$, we have $\mathfrak{a}' = \langle \mathfrak{a} \rangle_{k+r+1} + \langle d_1, \ldots, d_m \rangle_{k+r+1}$ where d_i is the defining relation of $s^{k+r+1} t_i$ $(i = 1, \ldots, m)$.

Consider the special case $\mathfrak{a}' = \langle \mathfrak{a} \rangle + \langle d_1 \rangle \trianglelefteq R_{k+r}[s^{k+r+1} t_1]$. Applying Lemma 5.3 to $\mathfrak{p}_i$, we obtain that $\mathfrak{p}_i + \langle d_1 \rangle$ is prime and its function field is isomorphic to that of $\mathfrak{p}_i$. The field isomorphism implies the claim about the dimension.

The general case $\mathfrak{a}' = \langle \mathfrak{a} \rangle_{k+r+1} + \langle d_1, \ldots, d_m \rangle_{k+r+1}$ is proven by repeating the argument m times. This is possible because d_i does not depend on variables $s^{k+r+1} t_j$ with $j > i$ by Def. 4.1.(2). □

Theorem 5.5. *Algorithm 5.1 terminates.*

Proof. The only critical part is the loop in lines 4–7. We define an ordering $\prec$ on ideals as follows. Let $\mathfrak{a} \trianglelefteq A$ be a nonzero ideal in some Noetherian ring A, and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the associated prime ideals of $\mathfrak{a}$, labeled such that $\dim \mathfrak{p}_i \geq \dim \mathfrak{p}_{i+1}$ $(i = 1, \ldots, s-1)$. Equip $\mathfrak{a}$ with the integer vector

$$v(\mathfrak{a}) := (\dim \mathfrak{p}_1, \dim \mathfrak{p}_2, \ldots, \dim \mathfrak{p}_s) \in \mathbb{N}^s.$$

Another ideal $\tilde{\mathfrak{a}} \trianglelefteq \tilde{A}$ is called smaller than $\mathfrak{a}$, $\tilde{\mathfrak{a}} \prec \mathfrak{a}$, whenever $v(\tilde{\mathfrak{a}})$ is lexicographically smaller than $v(\mathfrak{a})$, i.e., the leftmost entry of $v(\tilde{\mathfrak{a}})$ which differs from the corresponding entry of $v(\mathfrak{a})$ is smaller than this entry.

It suffices to show that $I_{k+1} \prec I_k$, because then the ideal sequence $I_1, I_2, \dots$ computed by the algorithm is strictly decreasing wrt. $\prec$, and hence, by Dickson's lemma [Cox et al. 1992], it must be finite. Eventually, there will be a k with $I_k = \langle 1\rangle$ and at least then the loop is left.

Suppose $s^{k+r}t_m \notin \operatorname{Rad} I_k$ at the end of the loop body, otherwise we are done. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated primes of I_k. Evidently $\dim(\mathfrak{p}_i + \langle s^{k+r}t_m\rangle) \leq \dim \mathfrak{p}_i$ in R_{k+r} for all i. Furthermore, by $s^{k+r}t_m \notin \operatorname{Rad} I_k$ there must be at least one $\mathfrak{p}_i$ with $s^{k+r}t_m \notin \mathfrak{p}_i$, and so $\dim(\mathfrak{p}_i + \langle s^{k+r}t_m\rangle_{k+r}) < \dim \mathfrak{p}_i$ in R_{k+r}. Hence $I_k + \langle s^{k+r}t_m\rangle_{k+r} \prec I_k$. Now, by

$$I_{k+1} = (\langle I_k\rangle_{k+r+1} + \langle s^{k+r}t_m\rangle_{k+r+1}) + \langle s^{k+1}D\rangle_{k+r+1},$$

Lemma 5.4 ensures that $v(I_k + \langle s^{k+r}t_m\rangle_{k+r}) = v(I_{k+1})$, and therefore $I_{k+1} \prec I_k$ as claimed. □

6. EXAMPLES

Example 6.1. *In order to have the algorithm iterate k times ($k \in \mathbb{N}$ fixed), consider $f(n) = (n-1)(n-2)\dots(n-k)$ represented via*

$$D = \{st_1 - t_1 - 1, t_2 - (t_1 - 1)\dots(t_1 - k), st_2 - (st_1 - 1)\dots(st_1 - k)\}.$$

Applying the algorithm to the query $f(n) \stackrel{?}{=} 0$ will force at least k iterations, because for any smaller value the algorithm would return "true," quite in contrast to the correct result. In view of this example, Theorem 5.5 asserts that $\mathcal{C}$ does not contain nonzero sequences having arbitrary long runs of zero.

Example 6.2. *(Example 4.4 continued) We apply Algorithm 5.1 to show*

$$f(n) = \frac{\mathrm{F}(n)}{\mathrm{F}(n+1)} + \sum_{k=1}^{n} \frac{(-1)^k}{\mathrm{F}(k)\,\mathrm{F}(k+1)} = 0$$

for all $n \in \mathbb{N}$. Using D from page 7 as set of defining relations, we find $sst_7 \notin \operatorname{Rad}\langle D \cup \{t_7, st_7\}\rangle_2$, $s^3t_7 \in \operatorname{Rad}\langle D \cup sD \cup \{t_7, st_7, sst_7\}\rangle_3$. We conclude $k = 1$ and we have to check $k+r = 3$ initial values. It is easily verified $f(1) = f(2) = f(3) = 0$, and this implies $f(n) = 0$ for all $n \in \mathbb{N}$.

A careful inspection of the proofs in Section 5 shows that condition (2) of Def. 4.1 is only used in the termination proof, but not needed for the correctness. If we apply Algorithm 5.1 to a set $D \subseteq R_r$ which satisfies conditions (1) and (3) of Def. 4.1 and we obtain an answer, then this result is correct — we may, however, obtain no answer at all. The next example is an application of this observation.

Example 6.3. *(from [Graham et al. 1994], Exercise 5.93) We want to show for all functions f and all $\alpha \neq 0$ the identity*

$$\sum_{k=1}^{n} \frac{\prod_{i=1}^{k-1}\big(f(i)+\alpha\big)}{\prod_{i=1}^{k} f(i)} = \frac{1}{\alpha}\Big(\prod_{k=1}^{n} \frac{f(k)+\alpha}{f(k)} - 1\Big).$$

The idea is to omit the defining relation for the variable corresponding to $f(n)$. We will use $K = \mathbb{Q}$ as field of constants. The constant α is considered as constant sequence with undetermined initial values.

We use the following set as a set of defining relations. The variable t_3 will correspond to $f(n)$.

$$
\begin{aligned}
D = \{ & st_1 - t_1, && (\, t_1 \sim \alpha \,) \\
& t_2 t_1 - 1, st_2 st_1 - 1, && (\, t_2 \sim 1/\alpha \,) \\
& st_4 - t_4 st_3, && (\, t_4 \sim \Pi f(n) \,) \\
& t_5 t_3 - 1, st_5 st_3 - 1, && (\, t_5 \sim 1/f(n) \,) \\
& st_6 - t_6(st_3 + st_1) st_5, && (\, t_6 \sim \Pi((f(n)+\alpha)/f(n)) \,) \\
& st_7 - t_7(t_3 + t_1), && (\, t_7 \sim \Pi(f(n)+\alpha) \,) \\
& t_8 t_4 - 1, st_8 st_4 - 1, && (\, t_8 \sim 1/\Pi(f(n)) \,) \\
& st_9 - t_9 - st_8 st_7, && (\, t_9 \sim \Sigma(\Pi/\Pi) \,) \\
& t_{10} - t_9 + t_2(t_6 - 1), && (\, t_{10} \sim \textit{identity candidate} \,) \\
& \quad st_{10} - st_9 + st_2(st_6 - 1) \,\}
\end{aligned}
$$

It is easily checked that $st_{10} \notin \mathrm{Rad}\langle D \cup \{t_{10}\}\rangle_1$ and

$$sst_{10} \in \mathrm{Rad}\langle D \cup sD \cup \{t_{10}, st_{10}\}\rangle_2.$$

The loop terminates with $k = 1$ and we have to check $k + r = 2$ initial values. For $n = 1$, the left hand side evaluates to $1/f(1)$, and the right hand side evaluates to

$$\frac{1}{\alpha}\Big(\prod_{k=1}^{1} \frac{f(k)+\alpha}{f(k)} - 1\Big) = \frac{1}{\alpha}\Big(\frac{f(1)+\alpha}{f(1)} - 1\Big) = \frac{1}{f(1)}.$$

For $n = 2$, the left hand side evaluates to

$$\sum_{k=1}^{2} \frac{\prod_{i=1}^{k-1}\big(f(i)+\alpha\big)}{\prod_{i=1}^{k} f(i)} = \frac{1}{f(1)} + \frac{f(1)+\alpha}{f(1)f(2)} = \frac{\alpha + f(1) + f(2)}{f(1)f(2)}.$$

The right hand side evaluates to

$$\frac{1}{\alpha}\Big(\prod_{k=1}^{2} \frac{f(k)+\alpha}{f(k)} - 1\Big) = \frac{1}{\alpha}\Big(\frac{(f(1)+\alpha)(f(2)+\alpha)}{f(1)f(2)} - 1\Big) = \frac{\alpha + f(1) + f(2)}{f(1)f(2)}.$$

This completes the proof.

It is possible to restore the termination property of the algorithm in situations like above by regarding the free difference variables as elements of the ground field, as pointed out in [Kauers 2004].

While the examples above were selected in order to illustrate the computations of the algorithm in detail, the next example lists some identities which we were able to check automatically using our algorithm. Note that all these identities were up to now out of the scope of algorithmic computer proofs.

Example 6.4. (*1*) *Exercise 6.61 in [Graham et al. 1994]. If* $\mathrm{F}(n)$ *denotes the nth Fibonacci number then*

$$\sum_{k=0}^{n} \frac{1}{\mathrm{F}(2^k)} = 3 - \frac{\mathrm{F}(2^n - 1)}{\mathrm{F}(2^n)}.$$

(2) *(5.1.45) in [Abramowitz and Stegun 1972]. Let $E_n(x)$ denote the nth exponential integral and $\Gamma(n,x)$ be the incomplete Gamma function (See Ex. 3.1 for definitions and defining relations). Then*

$$E_n(x) = x^{n-1}\Gamma(1-n,x)$$

(3) *(26.4.5) in [Abramowitz and Stegun 1972]. Let $Q(\chi^2|n)$ be the quantile of the Chi Square distribution (cf. Ex. 3.1), and let $n!! = 2\cdot 4\cdot 6\cdots n$ for $n\in\mathbb{N}$ even. Then*

$$Q(\chi^2|n) = \exp(-\chi^2/2)\big(1 + \sum_{r=1}^{n/2-1}\frac{(\chi^2)^r}{(2r)!!}\big).$$

(4) *Recall the notation $\mathrm{K}_{k=1}^n a_k$ introduced in Section 3 for continued fractions, and let*

$$h(n) = \frac{(n+1)((-1)^n(\pi+2)+\pi-2)\Gamma(\frac{n}{2})}{4\sqrt{\pi}\,\Gamma(\frac{n+1}{2})} \qquad (n\in\mathbb{N}).$$

Then

$$\sum_{k=0}^{n}\frac{1}{k!} = \mathop{\mathrm{K}}_{k=1}^{n}\Big(h(k) - \prod_{i=1}^{k}\big(-\tfrac{7}{4}i^2 + 9i - \tfrac{45}{4}\big)\Big).$$

7. SOME REMARKS ON COMPLEXITY ISSUES

We have no results about the time and space complexity of Algorithm 5.1. A complexity analysis would have to focus on two questions: First, how much time is consumed by the radical membership test in line 4, and secondly, how many iterations of the while loop might be necessary. Using the recurrence relations given in D, it is possible to evaluate $f(n)$ in a number of field operations that is linear in n, so the contribution of lines 8–10 may be neglected.

It is generally hard to make statements about the time complexity of algorithms in commutative algebra. It was mentioned that the radical membership test can be done by a standard application of Gröbner basis techniques, but the computation of Gröbner bases is known to be expensive: doubly exponential runtime and exponential space requirements have to be assumed in general for the worst case. The special problem of radical membership can, however, be decided with polynomial space [Mayr 1997].

As for the number of iterations, it is not likely that a reasonable bound depending on, say, m, r and the maximum total degree d of the polynomials in D could be established. In fact, any such bound $\kappa(m,r,d)$ would give rise to a much faster algorithm for deciding zero equivalence, provided that κ itself can be computed in reasonable time: it would be sufficient to evaluate $f(n)$ for $n = 1,\dots,\kappa(m,r,d)$, and if $f(n) = 0$ for all those n, then $f(n) = 0$ for all n, as follows directly from the proof of Theorem 5.2. The while loop with its expensive radical membership test could be discarded altogether.

Despite its conjectured poor worst case complexity, we want to stress that our algorithm performs quite well in practice. It is well known that Gröbner basis computations perform far better than suggested by the worst case complexity analysis on problems arising in practice. A similar remark applies to the number of iterations of the loop in lines 4–7. Though it is easy to construct input which requires any prescribed number of iterations, already two or three iterations suffice for most examples we tried. Indeed, all the examples given in Section 6 were completed with only a few seconds of CPU time (2.4GHz, 1Gb RAM) — at least if fast special purpose software is used for the Gröbner basis computations. We have used Faugère's Gb system [Faugère 2002] in Example 11.(3) and Example 11.(4), all other examples are also in the scope of Mathematica's builtin for computing Gröbner basis. This may underline the applicability of the algorithm to instances arising from practical applications.

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