Computer-Assisted Proofs of Some Identities for Bessel Functions of Fractional Order

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In memory of Frank W.J. Olver (1924-2013)

Abstract We employ computer algebra algorithms to prove a collection of identities involving Bessel functions with half-integer orders and other special functions. These identities appear in the famous Handbook of Mathematical Functions, as well as in its successor, the DLMF, but their proofs were lost. We use generating functions and symbolic summation techniques to produce new proofs for them.

1 Introduction

The Digital Library of Mathematical Functions [DLMF] is the successor of the classical Handbook of Mathematical Functions [AS73] by Abramowitz and Stegun. Beginning of June 2005 Peter Paule was supposed to meet Frank Olver, the mathematics editor of the [DLMF], at the NIST headquarters in Gaithersburg (Maryland, USA). On May 18, 2005, Olver sent the following email to Paule:

"The writing of DLMF Chapter BS[1] by Leonard Maximon and myself is now largely complete [...] However, a problem has arisen in connection with about a dozen formulas from Chapter 10 of Abramowitz and Stegun for which we have not yet tracked down proofs, and the author of this chapter, Henry Antosiewiecz, died about a year ago. Since it is the editorial policy for the DLMF not to state formulas

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[1] finally Chapter 10 Bessel Functions

without indications of proofs, I am hoping that you will be willing to step into the breach and supply verifications by computer algebra methods [...] I will fax you the formulas later today."

In view of the upcoming trip to NIST, Paule was hoping to be able to provide at least some help in this matter. But the arrival of Olver's fax chilled the enthusiasm quite a bit. Despite containing some identities with familiar pattern, the majority of the entries involved Bessel functions of fractional order or with derivatives applied with respect to the order.

Let us now display the bunch of formulas we are talking about. Here, $J_\nu(z)$ and $Y_\nu(z)$ denote the Bessel functions of the first and second kind, respectively, $I_\nu(z)$ and $K_\nu(z)$ the modified Bessel functions, $j_n(z)$ and $y_n(z)$ the spherical Bessel functions, $P_n(z)$ the Legendre polynomials, and $\mathrm{Si}(z)$ and $\mathrm{Ci}(z)$ the sine and cosine integral, respectively. Unless otherwise specified, all parameters are arbitrary complex numbers.

$$\frac{1}{z}\sin\sqrt{z^2+2zt} = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}y_{n-1}(z) \qquad (2|t|<|z|, |\Im(z)|\le\Re(z)) \tag{10.1.39}$$

$$\frac{1}{z}\cos\sqrt{z^2-2zt} = \sum_{n=0}^{\infty}\frac{t^n}{n!}j_{n-1}(z) \qquad z\neq 0 \tag{10.1.40}$$

$$\left[\frac{\partial}{\partial\nu}j_\nu(z)\right]_{\nu=0} = \frac{1}{z}(\mathrm{Ci}(2z)\sin z-\mathrm{Si}(2z)\cos z) \qquad (z\in\mathbb{C}\setminus]-\infty,0]) \tag{10.1.41}$$

$$\left[\frac{\partial}{\partial\nu}j_\nu(z)\right]_{\nu=-1} = \frac{1}{z}(\mathrm{Ci}(2z)\cos z+\mathrm{Si}(2z)\sin z) \qquad (z\in\mathbb{C}\setminus]-\infty,0]) \tag{10.1.42}$$

$$\left[\frac{\partial}{\partial\nu}y_\nu(z)\right]_{\nu=0} = \frac{1}{z}(\mathrm{Ci}(2z)\cos z+[\mathrm{Si}(2z)-\pi]\sin z) \qquad (z\in\mathbb{C}\setminus]-\infty,0]) \tag{10.1.43}$$

$$\left[\frac{\partial}{\partial\nu}y_\nu(z)\right]_{\nu=-1} = -\frac{1}{z}(\mathrm{Ci}(2z)\sin z-[\mathrm{Si}(2z)-\pi]\cos z) \qquad (z\in\mathbb{C}\setminus]-\infty,0]) \tag{10.1.44}$$

$$J_0(z\sin\theta) = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!^2}j_{2n}(z)P_{2n}(\cos\theta) \tag{10.1.48}$$

$$j_n(2z) = -n!z^{n+1}\sum_{k=0}^{n}\frac{2n-2k+1}{k!(2n-k+1)!}j_{n-k}(z)y_{n-k}(z) \quad (n=0,1,2,\dots) \tag{10.1.49}$$

$$\sum_{n=0}^{\infty}j_n^2(z) = \frac{\mathrm{Si}(2z)}{2z} \tag{10.1.52}$$

$$\frac{1}{z}\sinh\sqrt{z^2-2\mathrm{i}zt} = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}\sqrt{\tfrac{1}{2}\pi/z}I_{-n+\frac{1}{2}}(z) \qquad (2|t|<|z|, |\Im(z)|\le\Re(z)) \tag{10.2.30}$$

$$\frac{1}{z}\cosh\sqrt{z^2+2\mathrm{i}zt} = \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^n}{n!}\sqrt{\tfrac{1}{2}\pi/z}I_{n-\frac{1}{2}}(z) \qquad z\neq 0 \tag{10.2.31}$$

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=1/2} = -\frac{1}{\sqrt{2\pi z}}(\mathrm{Ei}(2z)\mathrm{e}^{-z}+\mathrm{E}_1(2z)\mathrm{e}^{z}) \qquad (z\in\mathbb{C}\setminus\,]-\infty,0]) \tag{10.2.32}$$

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=-1/2} = \frac{1}{\sqrt{2\pi z}}(\mathrm{Ei}(2z)\mathrm{e}^{-z}-\mathrm{E}_1(2z)\mathrm{e}^{z}) \qquad (z\in\mathbb{C}\setminus\,]-\infty,0]) \tag{10.2.33}$$

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=\pm 1/2} = \pm\sqrt{\frac{\pi}{2z}}\mathrm{E}_1(2z)\mathrm{e}^{z} \qquad (z\in\mathbb{C}\setminus\,]-\infty,0]) \tag{10.2.34}$$

The numbering follows that in Abramowitz and Stegun [AS73], and Olver remarked on the fax: "Irene Stegun left a record (without proofs) that (10.1.41)-(10.1.44) have errors: the factor $\frac{1}{2}\pi$ should not be there, and (10.1.44) also has the wrong sign. Equations (10.2.32)-(10.2.34) have similar errors. Their correct versions are given by [...]".

In view of these unfamiliar objects and of the approaching trip to NIST, Paule asked his young collaborators for help. Within two weeks, all identities succumbed to the members of the algorithmic combinatorics group of RISC. Moreover, in addition to the [AS73] typos mentioned by Olver, further typos in (10.1.39) and (10.2.30) were found. Above we have listed the corrected versions of the formulas, and when we use the numbering from [AS73], we refer to the corrected versions of the formulas here and throughout the paper.

At this place we want to relate the [AS73] numbering to the one used in the [DLMF]: (10.1.39) and (10.1.40) are DLMF entries 10.56.2 and 10.56.1, respectively. With the help of the rewriting rule DLMF 10.47.3, (10.1.41) and (10.1.42) are DLMF entries 10.15.6 and 10.15.7, respectively; using the rule DLMF 10.47.4, (10.1.43) and (10.1.44) are DLMF entries 10.15.8 and 10.15.9, respectively. Entry (10.1.48) is DLMF 10.60.10, (10.1.49) is DLMF 10.60.4, and (10.1.52) is DLMF 10.60.11. With the help of DLMF 10.47.8, entry (10.2.30) turns into DLMF 10.56.4; and with the help of DLMF 10.46.7, entry (10.2.31) turns into DLMF 10.56.3. Formulas (10.2.32) and (10.2.33) are bundled in DLMF entry 10.38.6; formula (10.2.34) is DLMF 10.38.7.

The goal of our exposition is to convince the reader that only a very limited amount of techniques has to be mastered to be able to prove such special function identities with computer algebra.

Our computer proofs are based on the algorithmic theory of holonomic functions and sequences, and symbolic summation algorithms. In the following two sections, we do purely algebraic manipulations; where necessary, analytical justifications (convergence of series, etc.) are given in Section 4. In general, we rely on the following computer algebra toolbox; underlying ideas are described in [KP11, K13].

Holonomic closure properties. The packages gfun [SZ94] (for Maple) and GeneratingFunctions [Mal96] (for Mathematica) are useful for the manipulation of functions $f(x)$ that satisfy linear ordinary differential equations (LODEs) with polynomial coefficients, as well as for sequences f_n satisfying linear recurrence equations (LOREs) with polynomial coefficients. Such objects are called *holonomic*. It can be shown that whenever $f(x)$ and $g(x)$ (resp. f_n and g_n) are holonomic, then so are $f(x)\cdot g(x)$ and $f(x)+g(x)$ (resp. $f_n \cdot g_n$ and $f_n + g_n$). Furthermore, if $f(x) = \sum_{n=0}^{\infty} f_n x^n$, then $f(x)$ is holonomic if and only if f_n is holonomic as a sequence. The packages gfun and GeneratingFunctions provide procedures for "executing closure properties," i.e., from given differential equations for $f(x)$ and $g(x)$ they can compute differential equations for $f(x)\cdot g(x)$ and $f(x)+g(x)$, and likewise for sequences. Also several further closure properties can be executed in this sense, and there are procedures for obtaining a recurrence equation for f_n from a differential equation for its generating function $f(x) = \sum_{n=0}^{\infty} f_n x^n$, and vice versa.

Symbolic summation tools. The package Zb [PS95] (for Mathematica) and the more general and powerful packages Mgfun [Chy00] (for Maple), HolonomicFunctions [Kou09] and Sigma [Sch05, Sch07] (both for Mathematica) provide algorithms to compute for a given definite sum $S(n,z) = \sum_{k=0}^{n} f(n,z,k)$ recurrences (in n) and/or differential equations (in z). Here the essential assumption is that the summand $f(n,z,k)$ satisfies certain types of recurrences or differential equations; see Section 3.

Subsequently, we restrict our exposition to the Mathematica packages GeneratingFunctions, Zb, HolonomicFunctions, and Sigma. In the Appendix, for the reader's convenience we list all formulas from Abramowitz and Stegun [AS73] that we apply in our proofs.

As for applications of differentiating Bessel functions w.r.t. order, we mention maximum likelihood estimation for the generalized hyperbolic distribution, and calculating moments of the Hartman-Watson distribution. Both distributions have applications in mathematical finance [Pra99, Ger11]. Prause's PhD thesis [Pra99] in fact cites formulas (9.6.42)–(9.6.46).

2 Basic Manipulations of Power Series

Let us now show how to apply these computer algebra tools for proving identities. The basic strategy is to determine algorithmically a differential equation (LODE) or a recurrence (LORE) for both sides of an identity and check initial conditions.

First we load the package GeneratingFunctions in the computer algebra system Mathematica.

```
In[1]:= << GeneratingFunctions.m
        GeneratingFunctions Package by Christian Mallinger – © RISC Linz
```

2.1 LODE and initial conditions for (10.1.39)

We show that both sides of the equation satisfy the same differential equation in t, and then check a suitable number of initial values.

First we compute a differential equation for the left hand side $\frac{1}{z}\sin\sqrt{z^2+2zt}$. We view this function as the composition of $\frac{1}{z}\sin(t)$ with $\sqrt{z^2+2zt}$ and compute a differential equation for it from defining equations of the components, by using the command AlgebraicCompose. (The last argument specifies the function under consideration. This symbol is used both in input and output.)

In[2]:= $\text{AlgebraicCompose}[f''[t] == -f[t], f[t]^2 == z^2+2zt, f[t]]$

Out[2]= $zf[t]+f'[t]+(2t+z)f''[t] == 0$

In order to obtain a differential equation for the right hand side, we first compute a recurrence equation for the coefficient sequence $c_n := (-1)^n/n!\,y_{n-1}(z)$ from the recurrences of its factors (using (10.1.19)). (The coefficient-wise product of power series is called *Hadamard product*, which explains the name of the command REHadamard.)

In[3]:= $\text{REHadamard}[c[n+1] == -c[n]/(n+1), c[n-1]+c[n+1] == (2(n-1)+1)/z\,c[n], c[n]]$
CanRE::denom : Warning. The input equation will be multiplied by its denominator.

Out[3]= $zc[n]+(1+n)(1+2n)c[n+1]+(1+n)(2+n)zc[n+2] == 0$

Then we convert the recurrence equation for c_n into a differential equation for its generating function $\sum_{n=0}^{\infty} c_n t^n$, which is the right hand side.

In[4]:= $\text{RE2DE}[\%, c[n], f[t]]$

Out[4]= $zf[t]+f'[t]+(2t+z)f''[t] == 0$

This agrees with output 2. To complete the proof, we need to check two initial values.

In[5]:= $\text{Series}[1/z\,\text{Sin}[\sqrt{z^2+2zt}], \{t,0,1\}]$

Out[5]= $\dfrac{\text{Sin}[\sqrt{z^2}]}{z}+\dfrac{\sqrt{z^2}\text{Cos}[\sqrt{z^2}]}{z^2}t+O[t]^2$

By (10.1.12) and (10.1.19), this agrees with the initial values of the right hand side for $z \in \mathbb{R}_{\geq 0}$. The extension to complex z will be discussed in Section 4.

Alternatively, we could have derived a differential equation only for the right hand side and then check with Mathematica that the left hand side satisfies this equation:

In[6]:= $\text{Out}[4]\ /.\ f \to (1/z\,\text{Sin}[\sqrt{z^2+2z\#}]\&)$

Out[6]= True

The proofs for (10.1.40), (10.2.30), and (10.2.31) follow the same scheme as the proof above. Both variants of the proof work in each case.

In summary, the most systematic way is to compute a differential equation for the difference of left hand side and right hand side, and then check that an appropriate number of initial values are zero.

2.2 *Proof of (10.1.41)*

This time we will not derive an LODE, but instead a recurrence relation for the Taylor coefficients of the difference of the left and the right hand side. The term $\log(z/2)$ that occurs in the pertinent expansion (9.1.64) is not analytic at $z=0$, hence we first treat that one "by hand." (Working with Taylor series at $z=1$, say, promises not much but additional complications.) This will leave us with a rather complicated expression for a holonomic formal power series, for which we have to prove that it is zero. At this point, we will employ the GeneratingFunctions package for computing a recurrence equation for the coefficient sequence of that series. Upon checking a suitable number of initial values, zero equivalence is then established.

One might think that we would not even have to compute the recurrences, since it is known a priori that the sum of two sequences satisfying recurrences of order r_1 and r_2, respectively, satisfies a recurrence of order at most r_1+r_2. The same holds for products, with $r_1 r_2$ instead of r_1+r_2. The catch is that the leading coefficient of the combined recurrence might have roots in the positive integers. It is clear that in order to give an inductive proof there must not be an integer root beyond the places where we check initial values.

Proposition 1. *Identity (10.1.41) holds for* $z \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$.

Proof. First we consider the left hand side. Using (10.1.1) and (9.1.64) from the Appendix, we get

$$\frac{\partial}{\partial \nu} j_\nu(z) = j_\nu(z) \log\frac{z}{2} - \frac{1}{2}\sqrt{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{\psi(\nu+n+\frac{3}{2})}{\Gamma(\nu+n+\frac{3}{2})}\frac{(\frac{1}{4}z^2)^n}{n!},$$

where $\Gamma(x)$ and $\psi(x)=\frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}$ denote the Gamma and digamma function, respectively. Hence, with (10.1.11),

$$\left[\frac{\partial}{\partial \nu} j_\nu(z)\right]_{\nu=0} = \frac{\sin z}{z}\log\frac{z}{2} - \frac{1}{2}\sqrt{\pi}\sum_{n=0}^{\infty}(-1)^n \frac{\psi(n+\frac{3}{2})}{\Gamma(n+\frac{3}{2})}\frac{(\frac{1}{4}z^2)^n}{n!}.$$

For the right hand side, we need (5.2.14), (5.2.16), and the Taylor expansions of $\sin z$ and $\cos z$. We have to show that

$$\begin{aligned}&\left[\frac{\partial}{\partial \nu} j_\nu(z)\right]_{\nu=0} - \big(\mathrm{Ci}(2z)\sin z - \mathrm{Si}(2z)\cos z\big)/z\\ &= \frac{\sin z}{z}\log\frac{z}{2} - \frac{1}{2}\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\psi(n+\frac{3}{2})}{\Gamma(n+\frac{3}{2})}\frac{(\frac{1}{4}z^2)^n}{n!} + \frac{\mathrm{Si}(2z)\cos z}{z}\end{aligned}$$

$$
\left.\begin{aligned}
&-\frac{\sin z}{z}\Big(\gamma+\log(2z)+\sum_{n=1}^{\infty}\frac{(-1)^n(2z)^{2n}}{2n(2n)!}\Big)\\
=\;&-\frac{1}{2}\sqrt{\pi}\sum_{n=0}^{\infty}\frac{(-1/4)^n\psi(n+\frac{3}{2})z^{2n}}{\Gamma(n+\frac{3}{2})n!}+2\sum_{n=0}^{\infty}\frac{(-4)^n z^{2n}}{(2n+1)(2n+1)!}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\\
&-\big(\gamma+2\log 2\big)\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n+1)!}+4z^2\sum_{n=0}^{\infty}\frac{(-4)^n z^{2n}}{2(n+1)(2(n+1))!}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n+1)!}
\end{aligned}\right\}\;(*)
$$

is identically zero, i.e., $c_n = 0$ for all $n \geq 0$, where c_n is defined as $(*) = \sum_{n=0}^{\infty} c_n z^{2n}$.

To this end, we compute step by step a recurrence equation for c_n from the various coefficient sequences appearing in $(*)$. We suppress some of the output, in order to save space. Recurrences for most of the inner coefficient sequences are easy to obtain. For instance, for

In[7]:= $f[n\_] := \dfrac{(-4)^n}{(2n+1)(2n+1)!}$

we have

In[8]:= $\text{FullSimplify}[f[n+1]/f[n]]$

Out[8]= $\dfrac{-2(2n+1)}{(n+1)(2n+3)^2}$

and hence the recurrence $f_{n+1} = \frac{-(2n+1)}{2(n+1)(2n+3)^2} f_n$. Only the series involving $\psi(n+3/2)$ requires a bit more work. Here, we use the package GeneratingFunctions to obtain a recurrence from the recurrence (6.3.5) for $\psi(n+3/2)$ and the first order recurrence of $(-1/4)^n/\Gamma(n+\frac{3}{2})n!$.

In[9]:= $\text{recSum} = \text{REHadamard}[f[n+1] == f[n] + \dfrac{1}{n+3/2},$
$\qquad f[n+1] == \dfrac{-1}{2(2n+3)(n+1)} f[n], f[n]];$

Next, we compute recurrence equations for the coefficient sequence of the two series products in $(*)$.

In[10]:= $\text{recSiCos} = \text{RECauchy}[f[n+1] == \dfrac{-2(2n+1)}{(n+1)(2n+3)^2} f[n],$
$\qquad f[n+1] == \dfrac{-1}{2(2n+1)(n+1)} f[n], f[n]];$

In[11]:= $\text{recCiSin} = \text{RECauchy}[f[n+1] == \dfrac{-2(n+1)}{(n+2)^2(2n+3)} f[n],$
$\qquad f[n+1] == \dfrac{-1}{2(n+1)(2n+3)} f[n], f[n]];$

The latter recurrence has to be shifted by 1, owing to the factor z^2.

In[12]:= $\text{recCiSin} = \text{recCiSin} \;/.\; f[n\_] \to f[n+1] \;/.\; n \to n-1;$

The recurrences collected so far can now be combined to a recurrence for c_n.

In[13]:= $\text{rec1} = \text{REPlus}[\text{recSiCos}, \text{recSum}, f[n]];$

In[14]:= $\text{rec2} = \text{REPlus}[\text{recCiSin}, f[n+1] == \dfrac{-1}{2(n+1)(2n+3)} f[n], f[n]];$

In[15]:= $\text{rec} = \text{REPlus}[\text{rec1}, \text{rec2}, f[n]]$

Out[15]= $5184(227+60n)f[n]+\cdots$

$$\cdots+7600(4+n)(5+n)(6+n)^2(9+2n)(11+2n)(13+2n)^2(167+60n)f[n+6]=0$$

The precise shape of the recurrence is irrelevant, it only matters that it has order 6 and that the coefficient of $f[n+6]$ (i.e., of c_{n+6}) does not have roots at nonnegative integers. As this is the case, we can complete the proof by checking that the coefficients of $z^0,\ldots,z^{10}$ in $(*)$ vanish, which can of course be done with Mathematica.

Alternatively, a similar proof can be obtained more conveniently using the package

In[16]:= << HolonomicFunctions.m

HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.6 (12.04.2012)

One of the main features of this package is the Annihilator command; it analyzes the structure of a given expression and executes the necessary closure properties automatically, in order to compute a system of differential equations and/or recurrences for the expression. We apply it to $(*)$:

In[17]:= $$\text{Annihilator}\Big[-\frac{\text{Sin}[z]}{z}\Big(\text{EulerGamma}+2\,\text{Log}[2]+\text{Sum}\Big[\frac{(-1)^n\,(2z)^{2n}}{2n\,(2n)!},\{n,1,\infty\}\Big]\Big) -\frac{\sqrt{\pi}}{2}\text{Sum}\Big[\frac{(-1/4)^n\,z^{2n}\,\text{PolyGamma}[0,n+3/2]}{n!\,\text{Gamma}[n+3/2]},\{n,0,\infty\}\Big]+\frac{\text{Cos}[z]}{z}\text{SinIntegral}[2z], \text{Der}[z]\Big]$$

Out[17]= $$\{(48z^5+95z^3)D_z^8+(864z^4+1900z^2)D_z^7+(576z^5+5436z^3+10830z)D_z^6+ (7968z^4+23684z^2+17100)D_z^5+(1440z^5+32442z^3+77002z)D_z^4+ (13344z^4+59332z^2+83448)D_z^3+(1344z^5+33596z^3+82858z)D_z^2+ (6240z^4+31404z^2+46892)D_z+(432z^5+6495z^3+15150z)\}$$

Since the HolonomicFunctions package uses operator notation, the second argument indicates that a differential equation w.r.t. z is desired; instead of an equation the corresponding operator is returned with $D_z=\mathrm{d}/\mathrm{d}z$. As before, the proof is completed by checking a few initial values (see also Section 4). □

3 Symbolic Summation Tools

It is not always the case that recurrences for the power series coefficients can be obtained by the package GeneratingFunctions. Sometimes combinatorial identities such as the following one are needed. Its proof gives occasion to introduce the Mathematica package Zb, an implementation of Zeilberger's algorithm for hypergeometric summation [Zei91].

Lemma 1. *For $k\in\mathbb{Z}_{\geq 0}$ we have*

$$\sum_{j=1}^{k}\frac{(-2)^j}{j}\binom{k}{j}=\begin{cases}\mathrm{H}_{n+1}-2\mathrm{H}_{2n+2} & k=2n+1 \quad \textit{is odd}\\ \mathrm{H}_n-2\mathrm{H}_{2n} & k=2n \quad \textit{is even,}\end{cases}$$

where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$ *denotes the harmonic numbers.*

It can be a chore to locate such identities in the literature. The closest match that the authors found is the similar identity $\sum_{j=1}^{k}(-1)^{j+1} j^{-1}\binom{k}{j} = \mathrm{H}_k$ [GKP94, p. 281]. Thus, an automatic identity checker like the one we describe now is helpful. We note in passing that we can not only verify such identities, but even compute the right hand side from the left hand side [Sch05].

Proof (of Lemma 1). We denote the sum on the left hand side by a_k. Using the Mathematica package

In[18]:= << Zb.m

Fast Zeilberger Package by Peter Paule and Markus Schorn (enhanced by Axel Riese) – © RISC Linz

we find

In[19]:= $\mathrm{Zb}[(-2)^j/j\,\mathrm{Binomial}[2n+1,j],\{j,1,2n+1\},n]$

If '1 + 2 n' is a natural number, then:

Out[19]= $\{(n+1)(2n+3)\mathrm{SUM}[n]-(4n^2+14n+13)\mathrm{SUM}[n+1]$
$+(n+2)(2n+5)\mathrm{SUM}[n+2]==-2\}$

In[20]:= $\mathrm{Zb}[(-2)^j/j\,\mathrm{Binomial}[2n,j],\{j,1,2n\},n]$

If '2n' is a natural number, then:

Out[20]= $\{(n+1)(2n+1)\mathrm{SUM}[n]-(4n^2+10n+7)\mathrm{SUM}[n+1]$
$+(n+2)(2n+3)\mathrm{SUM}[n+2]==-2\}$

hence the sequence a_k satisfies the recurrences

$$(n+1)(2n+3)a_{2n+1}-(4n^2+14n+13)a_{2n+3}+(n+2)(2n+5)a_{2n+5}=-2$$

and

$$(n+1)(2n+1)a_{2n}-(4n^2+10n+7)a_{2n+2}+(2n+3)(n+2)a_{2n+4}=-2.$$

The right hand side satisfies these recurrences, too:

In[21]:= $\mathrm{Out}[19]$ /. $\mathrm{SUM}[n\_] \to \mathrm{HarmonicNumber}[n+1]-2\mathrm{HarmonicNumber}[2n+2]$
// ReleaseHold // FullSimplify

Out[21]= {True}

In[22]:= $\mathrm{Out}[20]$/. $\mathrm{SUM}[n\_] \to \mathrm{HarmonicNumber}[n]-2\mathrm{HarmonicNumber}[2n]$
// ReleaseHold // FullSimplify

Out[22]= {True}

Hence the desired result follows by checking the initial conditions $k=0,1,2,3$. □

Proposition 2. *Identities (10.2.32) and (10.2.33) follow from Lemma 1. They hold for* $z \in \mathbb{C}\setminus\mathbb{R}_{\le 0}$.

Proof. We do Taylor series expansion on both sides of (10.2.32), and then compare coefficients. Using the expansions (5.1.10) and (5.1.11) and computing Cauchy products, we find that the right hand side of (10.2.32) equals

$$\sqrt{\frac{2}{\pi z}}\left((\log z+\log 2+\gamma)\sinh z+\sum_{n=0}^{\infty}\frac{a_{2n+1}}{(2n+1)!}z^{2n+1}\right), \tag{1}$$

where a_k is the sum from Lemma 1. The expansion of the left hand side of (10.2.32) can be done with (9.6.10) and (9.6.42). Since

$$\frac{(z^2/4)^n}{\Gamma(n+\frac{3}{2})n!}=\frac{2z^{2n}}{\sqrt{\pi}(2n+1)!}$$

and

$$\psi(n+\tfrac{3}{2})=-\gamma-2\log 2+2\mathrm{H}_{2n+2}-\mathrm{H}_{n+1},$$

the left hand side of (10.2.32) turns out to be

$$\sqrt{\frac{2}{\pi z}}\left((\log z+\log 2+\gamma)\sinh z+\sum_{n=0}^{\infty}(\mathrm{H}_{n+1}-2\mathrm{H}_{2n+2})\frac{z^{2n+1}}{(2n+1)!}\right). \tag{2}$$

Lemma 1 completes the coefficient comparison.

Identity (10.2.33) can be proved analogously; replace sinh by cosh and $2n+1$ by $2n$ in (1), and sinh by cosh and the summand by $(\mathrm{H}_n-2\mathrm{H}_{2n})z^{2n}/(2n)!$ in (2). □

We proceed to prove the identities (10.1.48), (10.1.49), and (10.1.52) by the same strategy as above: compute LODEs or LOREs for both sides, and check initial values. Since in these identities definite sums occur for which one cannot derive LOREs or LODEs by using holonomic closure properties, symbolic summation algorithms enter the game. For hypergeometric sums, like in Lemma 1, the package Zb is the perfect choice. Since in the following identities the occurring sums do not have hypergeometric summands, we use more general summation methods [Sch05] and [Kou09] that are available in the packages Sigma and HolonomicFunctions, respectively.

In general, the sums under consideration are of the form

$$S(n,z)=\sum_{k=0}^{\infty}h(n,k)f(n,z,k) \tag{3}$$

with integer parameter n and complex parameter z where h and f have the following properties: $h(n,k)$ is a hypergeometric term in n and k, i.e., $h(n+1,k)/h(n,k)$ and $h(n,k+1)/h(n,k)$ are rational functions in n and k. Furthermore, $f(n,z,k)$ satisfies a recurrence relation of the form

$$\begin{aligned}f(n,z,k+d)&=\alpha_0(n,z,k)f(n,z,k)\\&\quad+\alpha_1(n,z,k)f(n,z,k+1)+\cdots+\alpha_{d-1}(n,z,k)f(n,z,k+d-1),\end{aligned} \tag{4}$$

and either a recurrence relation

$$f(n+1,z,k)=\beta_0(n,z,k)f(n,z,k)$$

$$+\beta_1(n,z,k)f(n,z,k+1)+\cdots+\beta_{d-1}(n,z,k)f(n,z,k+d-1) \quad (5)$$

or a differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}f(n,z,k)=\beta_0(n,z,k)f(n,z,k)$$
$$+\beta_1(n,z,k)f(n,z,k+1)+\cdots+\beta_{d-1}(n,z,k)f(n,z,k+d-1), \quad (6)$$

where the α_i,β_i are rational functions in k, n, and z. From recurrences of the forms (4) and (5) we will derive a recurrence relation in n for $S(n,z)$. If, on the other hand, we have (6) instead of (5), we will compute a differential equation for $S(n,z)$ in z.

We note that the HolonomicFunctions package allows more flexible recurrence/differential systems as input specifying the shift/differential behavior of the summand accordingly. However, the input description given above gives rise to rather efficient algorithms implemented in the Sigma package to calculate LOREs and LODEs for $S(n,z)$.

3.1 LORE and initial conditions for (10.1.49)

We compute a LORE for the right hand side

$$S(n):=\sum_{k=0}^{n}-n!z^{n+1}\frac{2n-2k+1}{k!(2n-k+1)!}j_{n-k}(z)y_{n-k}(z)$$
$$=\sum_{k=0}^{n}\frac{-n!z^{n+1}(2k+1)}{(n-k)!(n+k+1)!}j_k(z)y_k(z)$$

using

In[23]:= $<<$ Sigma.m

Sigma - A summation package by Carsten Schneider © RISC-Linz

First we insert the sum in the form (3) with recurrences of the type (4) and (5). Note that $h(n,k)=\frac{-n!z^{n+1}(2k+1)}{(n-k)!(n+k+1)!}$ is hypergeometric in n and k. Moreover, by (10.1.19) the spherical Bessel functions of the first kind $j(k):=j_k(z)$ (we suppress the parameter z in our Mathematica session) fulfill the recurrence

In[24]:= $\mathrm{recJ}=z\,j[k]-(2k+3)\,j[k+1]+z\,j[k+2]==0;$

Since the same recurrence holds for $y_k(z)$, see (10.1.19), we obtain with

In[25]:= $\mathrm{recJY}=\mathrm{REHadamard}[\mathrm{recJ},\mathrm{recJ},\mathrm{j}[k]]/.\{\mathrm{j}\to\mathrm{f}\};$

Out[25]= $(-2k-5)z^2\mathrm{f}[k]+(2k+3)(4k^2+16k-z^2+15)f[k+1]$
$-(2k+5)(4k^2+16k-z^2+15)f[k+2]+(2k+3)z^2f[k+3]=0$

a recurrence in the form (4) for $f(k) := j_k(z)y_k(z)$. Since $f(k)$ is free of n, we choose $f[n+1,k] == f[k]$ for the required recurrence of the form (5). Given these recurrences we are ready to compute a recurrence for our sum

In[26]:= $\text{mySum} = \sum_{k=0}^{n} \frac{-n!z^{n+1}(2k+1)}{(n-k)!(n+k+1)!}\text{f}[k];$

by using the Sigma-function

In[27]:= $\text{GenerateRE}[\text{mySum}, n, \{\text{recJY}, f[k]\}, f[n+1,k] == f[k]]$

Out[27]= $2z\text{SUM}[n] - (2n+3)\text{SUM}[n+1] + 2z\text{SUM}[n+2] == 0$

Note that $S(n) = \sum_{k=0}^{n} h(n,k)f(k)(= \text{mySum} = \text{SUM}[n])$. Since besides $S(n)$ also $j_n(2z)$ fulfills the computed recurrence and since $S(n) = j_n(2z)$ for $n = 0,1$, we have $S(n) = j_n(2z)$ for all $n \geq 0$.

A correctness proof. Denote $\Delta_k g(z,k) := g(z,k+1) - g(z,k)$. The correctness of the produced recurrence follows from the computed proof certificate

$$\Delta_k g(n,k) = c_0 h(n,k)f(k) + c_1 h(n+1,k)f(k) + c_1 h(n+1)f(k) \tag{7}$$

given by $c_0(n) = 2z$, $c_1(n)$, $c_2 = -(2n+3)$, $c_2 = 2z$ and

$$g(n,k) = \frac{z^{n+1}n!}{(2k+3)(n+k+2)!(n-k+2)!}\left[g_0\text{f}(k) + g_1\text{f}(k+1) + g_2\text{f}(k+2)\right]$$

with

$$\begin{aligned} g_0 =& 8k^5 - 8(n-1)k^4 - (z^2+28n+30)k^3 + 2(2n^2+(2z^2-9)n+2z^2-19) \\ & k^2 + ((z^2+8)n^2 + (8z^2+15)n + 8z^2+1)k + (n^2+3n+2)(2z^2+3) \\ g_1 =& (2k+3)(k-n-2)(2k^3+(3-2n)k^2-(5n+2)k+(n+1)(z^2-3)), \\ g_2 =& -(k+1)(k-n-2)(k-n-1)z^2. \end{aligned}$$

Namely, one can show that (7) holds for all $n \geq 0$ and $0 \leq k \leq n$ as follows. Express $\Delta_k g(n,k)$ in terms of $f(k)$ and $f(k+1)$ by using the recurrence given in Out[25] and rewrite any factorial in (7) in terms of $(n+k+2)!$ and $(n-k+2)!$. Afterwards verify (7) by polynomial arithmetic. The summation of (7) over k from 0 to n gives the recurrence in Out[27]; here we needed the first evaluations of $f(i) = j_i(z)y_i(z)$, $i = 0,1,2$, from (10.1.11) and (10.1.12).

We remark that the underlying algorithms [Sch05] unify the creative telescoping paradigm [Zei91] in the difference field setting [Sch07] and holonomic setting [Chy00]. This general point of view opens up interesting applications, e.g., in the field of combinatorics [APS05] and particle physics [ABRS12].

3.2 LODE and initial conditions for (10.1.48)

For the of proof (10.1.48) we choose the package HolonomicFunctions. As we have seen before, holonomic closure properties include algebraic substitution; but since $\sin(\theta)$ is not algebraic, we have to transform identity (10.1.48) slightly in order to make it accessible to our software: just replace $\cos(\theta)$ by c and $\sin(\theta)$ by $\sqrt{1-c^2}$. Now it is an easy task to compute a LODE in z for the left hand side:

In[28]:= $\text{Annihilator}\left[\text{BesselJ}\left[0, z\sqrt{1-c^2}\right], \text{Der}[z]\right]$

Out[28]= $\left\{zD_z^2 + D_z + (z - c^2 z)\right\}$

The sum on the right hand side requires some more work. Similar to identity (10.1.49) above, the technique of creative telescoping [Zei91] is applied and it fits perfectly to the HolonomicFunctions package. The latter can deal with multivariate holonomic functions and sequences, i.e., roughly speaking, mathematical objects that satisfy (for each variable in question) either a LODE or a LORE of arbitrary (but fixed) order. For example, the expression

$$f(n,z,c) = (4n+1)\frac{(2n)!}{2^{2n}n!^2} j_{2n}(z) P_{2n}(c)$$

satisfies a LORE in n of order 4 and LODEs w.r.t. z and c, both of order 2. To derive a LODE in z for the sum we employ the following command (the shift operator S_n, defined by $S_n f(n) = f(n+1)$, is input as S[n], and the derivation D_z, defined by $D_z f(z) = f'(z)$, is input as Der[z]):

In[29]:= $\text{CreativeTelescoping}[(4n+1)(2n)!/(2^{2n}n!^2)\,\text{SphericalBesselJ}[2n,z]\,\text{LegendreP}[2n,c],$
$\text{S}[n]-1, \text{Der}[z]]$

Out[29]= $$\left\{\left\{zD_z^2 + D_z + (z - c^2 z)\right\}, \left\{\frac{4(n+1)^2}{4n+5} S_n D_z + \frac{4(n+1)^2(8n^2+18n-z^2+9)}{(4n+3)(4n+5)z} S_n + \frac{4n^2}{4n+1} D_z + \frac{-16c^2n^2z^2 - 16c^2nz^2 - 3c^2z^2 + 32n^4 + 40n^3 + 4n^2z^2 + 12n^2 + 4nz^2 + z^2}{(4n+1)(4n+3)z}\right\}\right\}$$

The output consists of two operators, say P and Q, which are called *telescoper* and *certificate* (note already that P equals Out[28]). They satisfy the relation

$$\big(P + (S_n - 1)Q\big)\, f(n,z,c) = 0, \tag{8}$$

a fact that can be verified using the well-known LODEs and LOREs for spherical Bessel functions and Legendre polynomials. Summing (8) w.r.t. n and telescoping yields

$$P\sum_{n=0}^{\infty} f(n,z,c) - (Qf)(0,z,c) + \lim_{n\to\infty}(Qf)(n,z,c) = 0$$

(P is free of n and S_n and therefore can be interchanged with the summation quantifier). Using (9.3.1) and (10.1.1) it can be shown that the limit is 0, and also the part $(Qf)(0,z,c)$ vanishes.

Consequently, we have established that both sides of (10.1.48) satisfy the same second-order LODE. It suffices to compare the initial conditions at $z=0$ (see Section 4). For the left hand side we have $J_0(0)=1$. From (10.1.25) it follows that the Taylor expansion of $j_{2n}(z)$ starts with z^{2n} and hence for $z=0$ all summands are zero except the first one. With (10.1.11) we see that the initial conditions on both sides agree.

Before turning to the next identity, we want to point to [Sch07] where a different computer algebra proof of (10.1.48) has been given. More examples of proving special function identities with the HolonomicFunctions package are collected in [KM11].

3.3 LODE and initial conditions for (10.1.52)

Again we compute a LODE with Sigma. In order to get a LODE of the left hand side of (10.1.52) we compute a LODE of its truncated version

In[30]:= $\text{mySum} = \sum_{k=0}^{a} j[k]^2;$

Note that the summand of our input-sum depends non-linearly on $j_k(z)$. In order to handle this type of summation input, Sigma needs in addition the package [Ger02]

In[31]:= << OreSys.m

OreSys package by Stefan Gerhold © RISC-Linz

for uncoupling systems of LODE-systems. Then using a new feature of Sigma we can continue as "as usual". Given the difference-differential equation of the form (6) for $j(k) := j_k(z)$ and $j^{(0,1)}(k,z) := \frac{\mathrm{d}}{\mathrm{d}z} j_k(z)$:

In[32]:= $\text{recZ} = j^{(0,1)}[k,z] == \frac{k}{z} j[k] + j[k+1];$

see (10.2.20), and the recurrence In[24] of the form (5), we compute a LODE for $\text{mySum}(=\text{SUM}[n])$:

In[33]:= $\text{mySum} = \sum_{k=0}^{a} j[k]^2;$

In[34]:= $\text{GenerateDE}[\text{mySum}, n, \{\text{recJ}, j[k]\}, \text{recZ}]$

Out[34]= $z\text{SUM}'[z] + \text{SUM}[z] == (zj[a]j[a+1] - (2a+1)j[a]^2) - (zj[0]j[1] - j[0]^2)$

A correctness proof. The correctness of the LODE can be checked by the computed proof certificate

$$\Delta_k g(z,k) = c_0 j(k)^2 + c_1 j^{(0,1)}(k,z)^2 \tag{9}$$

with $c_0 = 1$, $c_2 = z$ and $g(z,k) = zj(k)j(k+1) - (2k+1)j(k)^2$. Namely, one can easily show that (9) holds for all $0 \le k$ as follows. Express (9) in terms of $j(k)$ and $j(k+1)$ by using the recurrence given in In[24] and the difference-differential equation given in In[32]. Afterwards verify (9) by polynomial arithmetic. Then summing (9) over k from 0 to a gives the recurrence in Out[34]; here we used the initial values (10.1.11).

Next, we let $a \to \infty$. Then $j_a(z)$ tends to zero by (9.3.1). Therefore, the left hand side of (10.1.52) satisfies the LODE

$$S(z) + z\frac{\mathrm{d}S(z)}{\mathrm{d}z} = \frac{\sin(2z)}{2z}. \tag{10}$$

It is readily checked that the right hand side satisfies it, too, and both sides equal 1 at $z = 0$. This establishes equality of both sides of (10.1.52).

Alternatively, we can derive the inhomogeneous differential equation for the left hand side of (10.1.52) with HolonomicFunctions:

In[35]:= Annihilator[Sum[SphericalBesselJ[n, z]^2, $\{n, 0, \text{Infinity}\}$], Der[$z$], Inhomogeneous $\to$ True]

Out[35]= $\{\{zD_z + 1\}, \{\text{Hold}[\text{Limit}[\ldots, n \to \infty]] + \ldots\}\}$

The output consists of a differential operator and an expression that gives the inhomogeneous part (abbreviated above). Without help, Mathematica is not able to simplify the latter (i.e., compute the limit), but using (9.3.1) it succeeds and we get

$$\big(zD_z + 1\big)S(z) - \frac{\sin(z)\cos(z)}{z} = 0$$

which of course agrees with (10).

4 Series Solutions of LODEs and Analyticity

In some proofs we have determined a differential equation that is satisfied by both sides of the identity in question, and then compared initial values. In contrast to the case of recurrences, the validity of this approach needs some non-trivial justification. This procedure can be justified by well-known uniqueness results for solutions of LODEs, to be outlined in this section. In the proofs of (10.1.39), (10.1.40), (10.2.30), and (10.2.31), the point $t = 0$, where we checked initial conditions, is an ordinary point of the LODE (i.e., the leading coefficient of the LODE does not vanish at $t = 0$). Then there is a unique analytic solution, if the number of prescribed initial values equals the order of the equation. The identity then holds (at least) in the domain (containing zero) where we can establish analyticity of both sides.

Proposition 3. *Identity (10.1.40) holds for all complex t and all complex $z \neq 0$. The same is true for (10.2.31).*

Proof. We consider (10.1.40) and omit the analogous considerations for (10.2.31). For $n \in \mathbb{Z}$, the function $j_{n-1}(z)$ is defined for $z \in \mathbb{C}^*$. We fix such a z and consider both sides of (10.1.40) as functions of t. By (9.3.1), the right hand side converges uniformly for all complex t, therefore it is an entire function of t. The left hand side is also entire, since $\cos\sqrt{w} = \sum_{n\geq 0}(-1)^n w^n/(2n)!$ is an entire function of w. Initial values at $t = 0$ and an LODE satisfied by both sides were already presented in Section 2, hence, by the above uniqueness property, identity (10.1.40) is proved. $\square$

Proposition 4. *Identity (10.1.39) holds for all complex z and t with $|\Im(z)| \leq \Re(z)$ and $2|t| < |z|$. If $|\Im(z)| \leq -\Re(z)$, then the identity holds with switched sign for all t with $2|t| < |z|$. The same is true for (10.2.30).*

Proof. We give the proof in the case of (10.1.39); (10.2.30) is treated analogously. First we complete the check of initial values from Section 2. For $t = 0$, the right hand side is $y_{-1}(z) = (\sin z)/z$, and on the left hand side we have $(\sin\sqrt{z^2})/z$. Thus, at $t = 0$ both sides agree for $|\arg(z)| < \pi/2$, which follows from $|\Im(z)| \leq \Re(z)$; for $\pi/2 < |\arg(z)| < \pi$, which follows from $|\Im(z)| \leq -\Re(z)$, the identity holds at $t = 0$ with switched sign, because the function $w \mapsto \sqrt{w^2}$ changes sign when crossing the branch cut $\mathrm{i}\mathbb{R}$. The first derivatives at $t = 0$ are $(\cos\sqrt{z^2})/\sqrt{z^2} = (\cos z)/\sqrt{z^2}$ and $-y_0(z) = (\cos z)/z$, respectively. The same consideration as for the first initial value completes the check of the initial conditions.

Now we show that both sides of (10.1.39) are analytic functions of t for fixed $z \neq 0$ with $|\Im(z)| \leq |\Re(z)|$. Let us start by determining the radius of convergence of the right hand side. It is an easy consequence of (9.3.1) that

$$y_n(z) \sim -\frac{\sqrt{2}}{z}\left(\frac{2n}{\mathrm{e}z}\right)^n, \quad n \to \infty,\ z \neq 0.$$

Hence, by Stirling's formula, the radius of convergence is $|z|/2$, and so the right hand side is analytic for $2|t| < |z|$.

The left hand side of (10.1.39) has a branch cut along a half-line starting at $t = -z/2$, a point on the circle of convergence of the right hand side. If this half line has no other intersection with this circle, then the left hand side is analytic in the disk $\{t : 2|t| < |z|\}$. Otherwise, the branch cut separates the disk into two segments, and the identity does not necessarily hold in a segment that does not contain $t = 0$. As we will now show, our assumptions exclude the possibility of a second intersection. Once again it is convenient to proceed by computer algebra. Note that the presence of two intersections means that

$$\Big(\exists s \neq t \in \mathbb{C}\Big)\Big(2|s| = |z| \wedge 2|t| = |z| \wedge z^2 + 2zs \in\,]-\infty, 0] \wedge z^2 + 2zt \in\,]-\infty, 0]\Big)$$

holds. Upon rewriting this formula with real variables, it can be simplified by Mathematica's Reduce command; the result – translated back into complex language – is the equivalent formula $|\Re(z)| < |\Im(z)|$. Summing up, under our assumptions on z the left hand side of (10.1.39) is analytic in the disk $\{t : 2|t| < |z|\}$. □

Now consider the LODE (10), which we want to employ to prove (10.1.52). The point $z = 0$ is not an ordinary point, so the question of uniqueness of the solution is more subtle. The origin is a regular singular point of (10), since the degree of the indicial polynomial

$$[z^0]p_s(z)^{-1}z^{s-\sigma}\mathscr{L}z^\sigma = \sigma + 1$$

agrees with the order $s = 1$ of the LODE. Here, $p_s(z) = z$ denotes the leading coefficient, and $\mathscr{L}$ the differential operator

$$\mathscr{L} := 1 + zD_z.$$

The following classical result [Inc26] describes the structure of a fundamental system at a regular singular point. See also the concise exposition in Meunier and Salvy [MS03].

Theorem 1. *Let $z = 0$ be a regular singular point of a homogeneous LODE of order s. Denote the roots of the indicial polynomial by $\sigma_1, \ldots, \sigma_s$, and let $m_1, \ldots, m_s$ be their multiplicities. Then the equation has a basis of s solutions*

$$z^{\sigma_i} \sum_{j=0}^{d_i} \log^j(z) \Phi_{ij}(z), \qquad 1 \le i \le s, \tag{11}$$

where $d_i < s$, and the $\Phi_{ij}(z)$ are convergent power series. Each of these solutions is uniquely defined by the coefficients of the s "monomials"

$$\bigcup_{i=1}^{s} \left\{ z^{\sigma_i}, z^{\sigma_i} \log z, \ldots, z^{\sigma_i} \log^{m_i - 1} z \right\}$$

in the series (11)*.*

Proposition 5. *Identity (10.1.52) holds for all $z \in \mathbb{C}$.*

Proof. We have shown in the preceeding section that both sides satisfy the LODE (10). As seen above, the indicial polynomial of the homogeneous equation $\mathscr{L} f = f + zf' = 0$ is $\sigma + 1$. Hence, by Theorem 1, a solution of $\mathscr{L} f = 0$ that has the form (11) is uniquely defined by the coefficient of z^{-1}. Hence the zero function is the only analytic solution of the homogeneous initial value problem $\mathscr{L} f = 0$, $f(0) = 0$. It is a trivial consequence that the inhomogeneous equation (10) cannot have more than one analytic solution with $f(0) = 1$. Therefore, (10.1.52) holds in a neighbourhood of $z = 0$. The left hand side of (10.1.52) is entire since it is a uniform limit of entire functions, and the right hand side is entire by (5.2.14). Thus, the identity holds in the whole complex plane by analytic continuation. □

Proposition 6. *Identity (10.1.48) holds for all complex z and θ.*

Proof. By the Laplace-Heine formula [Sze75, Theorem 8.21.1], $P_{2n}(\cos\theta)$ grows at most exponentially as $n \to \infty$. Together with (9.3.1) and $n(2n)!/(2^{2n} n!^2) = \mathrm{O}(\sqrt{n})$, this shows that the right hand side of (10.1.48) is an entire function of z and θ. In Section 3 we showed that both sides of (10.1.48) satisfy the differential equation $zf''(z) + f'(z) + z(1 - c^2) f(z) = 0$ (whose indicial equation is $\sigma^2 = 0$) and that the initial condition at z^0 agrees. The result follows from Theorem 1 and the fact that both sides are entire functions. □

5 Non-Computer Proofs

Some of our identities can be easily proved from some of the others, without using any software machinery. The computer proofs that we have in hand suffice for establishing the remaining identities (10.1.42), (10.1.43), (10.1.44), and (10.2.34) in this spirit. The reader should by now be convinced that, if desired, all of them can also be proved by the algorithmic methods we have presented.

Proposition 7. *Identities (10.1.42), (10.1.43), and (10.1.44) follow from (10.1.41). They hold for* $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Proof. Identities (10.1.42), (10.1.43), and (10.1.44) can be done analogously to (10.1.41), but we instead present (non-computer) deductions from (10.1.41). The derivative of Y_ν w.r.t. ν can be expressed in terms of J_ν, $J_{-\nu}$, and Y_ν, see (9.1.65) in the appendix. Note that $\cot(\nu+1/2)\pi$ vanishes for $\nu = 0, 1$. (9.1.65) thus yields

$$\left[\frac{\partial}{\partial\nu} y_\nu(z)\right]_{\nu=0} = \left[\frac{\partial}{\partial\nu} j_\nu(z)\right]_{\nu=-1} - \frac{\pi \sin z}{z}$$

and

$$\left[\frac{\partial}{\partial\nu} y_\nu(z)\right]_{\nu=-1} = -\left[\frac{\partial}{\partial\nu} j_\nu(z)\right]_{\nu=0} - \frac{\pi \cos z}{z}.$$

Therefore, we have a relation between the left hand sides of (10.1.42) and (10.1.43), and one between the left hand sides of (10.1.41) and (10.1.44). It is easy to verify that the respective right hand sides satisfy the same relations. Hence the assertion will be established once we show that (10.1.42) follows from (10.1.41). To this end, it suffices to show that the left hand sides of these identities satisfy

$$\frac{\partial}{\partial z}\left(z\left[\frac{\partial}{\partial\nu} j_\nu(z)\right]_{\nu=0}\right) - z\left[\frac{\partial}{\partial\nu} j_\nu(z)\right]_{\nu=-1} = -\frac{\sin z}{z}, \tag{12}$$

since once again it is easy to see that the right hand sides of (10.1.41) and (10.1.42) obey the same relation. By (9.1.64), the recurrence relations of Γ and ψ, and the duplication formula of Γ, the left hand side of (12) equals

$$\begin{aligned}
&\frac{\sin z}{z} - \sqrt{\pi}\sum_{k=0}^{\infty}(-\tfrac{1}{4})^k\left(\frac{\psi(k+\frac{3}{2})}{\Gamma(k+\frac{3}{2})/(k+\frac{1}{2})} - \frac{\psi(k+\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\right)\frac{z^{2k}}{k!}\\
=&\frac{\sin z}{z} - \sqrt{\pi}\sum_{k=0}^{\infty}(-\tfrac{1}{4})^k\frac{1}{\Gamma(k+\frac{1}{2})(k+\frac{1}{2})}\frac{z^{2k}}{k!}\\
=&\frac{\sin z}{z} - \sum_{k=0}^{\infty}(-\tfrac{1}{4})^k\frac{2^{2k+1}z^{2k}}{\Gamma(2k+2)} = -\frac{\sin z}{z}.
\end{aligned}$$

□

Proposition 8. *Identity (10.2.34) follows from (10.1.32) and (10.2.33). It holds for* $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Proof. Indeed, by (9.6.43) we have

$$\begin{aligned}
\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=\pm1/2} &= \frac{\pi}{2}\csc(\nu\pi)\left[\frac{\partial}{\partial\nu}I_{-\nu}(z)-\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=\pm1/2}\\
&= -\frac{\pi}{2}\csc(\nu\pi)\left(\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=\mp1/2}+\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=\pm1/2}\right)\\
&= \pm\sqrt{\frac{\pi}{2z}}\mathrm{e}^z\mathrm{E}_1(2z).
\end{aligned}$$

□

Finally, we note that (10.2.32), which was proved in Proposition 2, can be proved by hand from (10.1.41). Indeed, replacing z with iz in (9.1.64) makes the k-sum in (9.1.64) equal the k-sum in (9.6.42). Solving both relations for the k-sum allows to express $\frac{\partial}{\partial\nu}I_\nu(z)$ by $I_\nu(z)$, $J_\nu(\mathrm{i}z)$, and $\frac{\partial}{\partial\nu}J_\nu(\mathrm{i}z)$. Plugging in $\nu=\frac{1}{2}$, rewriting $\frac{\partial}{\partial\nu}J_\nu(\mathrm{i}z)$ with (10.1.41), and using the relations (5.2.21) and (5.2.23) between the exponential integral and the sine and cosine integrals gives (10.2.32). Analogously, (10.2.33) follows from (10.1.42).

Acknowledgements This work has been supported by the Austrian Science Fund (FWF) grants P20347-N18, P24880-N25, Y464-N18, DK W1214 (DK6, DK13) and SFB F50 (F5004-N15, F5006-N15, F5009-N15), and by the EU Network LHCPhenoNet PITN-GA-2010-264564.

Appendix: List of Relevant Table Entries

For the reader's convenience, we collect here all identities from Abramowitz, Stegun [AS73] that we have used.

$$\mathrm{Ei}(x)=\gamma+\ln x+\sum_{n=1}^{\infty}\frac{x^n}{n\,n!}\qquad (x>0)\tag{5.1.10}$$

$$E_1(z)=-\gamma-\ln z-\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n\,n!}\qquad (|\arg z|<\pi)\tag{5.1.11}$$

$$\mathrm{Si}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}\tag{5.2.14}$$

$$\mathrm{Ci}(x)=\gamma+\log x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2n(2n)!}\tag{5.2.16}$$

$$\mathrm{Si}(x)=\frac{1}{2\mathrm{i}}(E_1(\mathrm{i}z)-E_1(-\mathrm{i}z))+\frac{\pi}{2}\qquad (|\arg z|<\frac{\pi}{2})\tag{5.2.21}$$

$$\mathrm{Ci}(x)=-\frac{1}{2}(E_1(\mathrm{i}z)+E_1(-\mathrm{i}z))\qquad (|\arg z|<\frac{\pi}{2})\tag{5.2.23}$$

$$\psi(z+1)=\psi(z)+\frac{1}{z}\tag{6.3.5}$$

$$\frac{\partial}{\partial\nu}J_\nu(z) = J_\nu(z)\log(\tfrac{1}{2}z) - (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty}(-1)^k \frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)}\frac{(\frac{1}{4}z^2)^k}{k!} \tag{9.1.64}$$

$$\frac{\partial}{\partial\nu}\,Y_\nu(z) = \cot(\nu\pi)\Big(\frac{\partial}{\partial\nu}J_\nu(z) - \pi Y_\nu(z)\Big) \tag{9.1.65}$$

$$-\csc(\nu\pi)\frac{\partial}{\partial\nu}J_{-\nu}(z) - \pi J_\nu(z) \qquad (\nu \neq 0, \pm 1, \pm 2, \dots)$$

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\Big(\frac{\mathrm{e}z}{2\nu}\Big)^\nu, \quad Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}}\Big(\frac{\mathrm{e}z}{2\nu}\Big)^{-\nu} \; (\nu\to\infty) \tag{9.3.1}$$

$$I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)} \tag{9.6.10}$$

$$\frac{\partial}{\partial\nu}I_\nu(z) = I_\nu(z)\ln(\tfrac{1}{2}z) - (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty}\frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)}\frac{(\frac{1}{4}z^2)^k}{k!} \tag{9.6.42}$$

$$j_n(z) = \sqrt{\tfrac{1}{2}\pi/z}\,J_{n+\frac{1}{2}}(z), \quad y_n(z) = \sqrt{\tfrac{1}{2}\pi/z}\,Y_{n+\frac{1}{2}}(z) \tag{10.1.1}$$

$$j_0(z) = \frac{\sin z}{z}, \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \tag{10.1.11}$$

$$j_2(z) = \Big(\frac{3}{z^3} - \frac{1}{z}\Big)\sin z - \frac{3}{z^2}\cos z$$

$$y_0(z) = -j_{-1}(z) = -\frac{\cos z}{z}, \quad y_1(z) = j_{-2}(z) = -\frac{\cos z}{z^2}, \tag{10.1.12}$$

$$y_2(z) = -j_{-3}(z) = \Big(\frac{1}{z} - \frac{3}{z^2}\Big)\cos z - \frac{3}{z^2}\sin z$$

$$j_{n-1}(z) + j_{n+1}(z) = (2n+1)z^{-1}j_n(z) \qquad (n\in\mathbb{Z}) \tag{10.1.19}$$

$$y_{n-1}(z) + y_{n+1}(z) = (2n+1)z^{-1}y_n(z) \qquad (n\in\mathbb{Z})$$

$$j_n(z) = z^n\Big(-\frac{1}{z}\frac{\partial}{\partial z}\Big)^n\frac{\sin z}{z} \tag{10.1.25}$$

$$\frac{n+1}{z}\,j_n(z) + \frac{\mathrm{d}}{\mathrm{d}z}j_n(z) = j_{n-1}(z) \tag{10.2.20}$$

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