A Non-Holonomic Systems Approach to Special Function Identities

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ABSTRACT

We extend Zeilberger's approach to special function identities to cases that are not holonomic. The method of creative telescoping is thus applied to definite sums or integrals involving Stirling or Bernoulli numbers, incomplete Gamma function or polylogs, which are not covered by the holonomic framework. The basic idea is to take into account the dimension of appropriate ideals in Ore algebras. This unifies several extensions that have been presented earlier and provides algorithms for summation and integration in classes that had not been accessible to computer algebra before.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*

General Terms

Algorithms

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1. INTRODUCTION

In a classical article entitled "A holonomic systems approach to special functions identities" [16], Doron Zeilberger has shown that the theory of holonomic D-modules leads to algorithms for proving identities in large classes of special functions. In this setting, a function $f(x_1, \ldots, x_n)$ is represented by a system of linear differential equations with polynomial coefficients that annihilate it. The function is "holonomic" when it possesses two important properties: *(i)* besides its defining system, f can be specified by a *finite* number of initial conditions; *(ii)* the number of linearly independent functions among all $x_1^{m_1} \cdots x_n^{m_n} \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}(f)$ with $m_1 + \cdots + m_n + k_1 + \cdots + k_n \leq N$ grows like $O(N^n)$. The first property has the consequence that many operations on holonomic functions reduce to linear algebra. It leads to closure properties under sum, product, and specialization. The second one is related to the notion of holonomic D-modules and opens the way to algorithms for definite integration and summation. For this, Zeilberger developed a general method called *creative telescoping*, for which he gave two algorithms: one for the general holonomic case and a faster one in the hypergeometric case.

Originally, the notion of a holonomic system is only defined for differential systems, but there are several ways of extending it to systems of difference (or q-difference) equations. Among those, we prefer the use of Ore algebras [6]. There, the first property above corresponds to zero-dimensional ideals, which are called ∂-finite. The same closure properties (sum, product, specialization) hold and can be performed by Gröbner bases computation. Chyzak [5] extended Zeilberger's fast hypergeometric creative telescoping to all ∂-finite ideals, termination being guaranteed inside Zeilberger's holonomic class.

Another direction of extension concerns functions or sequences that cannot be defined by a holonomic system or even a ∂-finite ideal. Majewicz [12] gives an algorithm that is able to produce Abel's summation identity

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n$$

automatically and to find similar new identities. Kauers [9] gives a summation algorithm applicable to sums involving Stirling numbers and similar sequences defined by triangular recurrence equations. This algorithm finds, for instance,

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} S_2(n+1, k+1) = E_1(n, m),$$

where S_2 and E_1 refer to the Stirling numbers of second kind and the Eulerian numbers of first kind, respectively. A summation algorithm of Chen and Sun [4] is able to discover certain summation identities involving Bernoulli numbers B_n

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Operator	$\partial \cdot f(x)$	$x \cdot f(x)$
Differentiation $\frac{d}{dx}$	$f'(x)$	$xf(x)$
Shift S	$f(x+1)$	$xf(x)$
Difference Δ	$f(x+1)-f(x)$	$xf(x)$
q-Dilation Q	$f(qx)$	$xf(x)$
Continuous q-difference	$f(qx)-f(x)$	$xf(x)$
q-Differentiation $\delta^{(q)}$	$\frac{f(qx)-f(x)}{(q-1)x}$	$xf(x)$
q-Shift $S^{(q)}$	$f(x+1)$	$q^x f(x)$
Discrete q-difference $\Delta^{(q)}$	$f(x+1)-f(x)$	$q^x f(x)$
Eulerian operator Θ	$xf'(x)$	$xf(x)$
Mahlerian operator M	$f(x^b)$	$xf(x)$
Divided differences	$\frac{f(x)-f(a)}{x-a}$	$xf(x)$

Table 1: Some common Ore operators

Operator	σ	δ	∂x
Differentiation	Id	$\frac{d}{dx}$	$x\partial+1$
Shift S	S	0	$(x+1)\partial$
Difference	S	Δ	$(x+1)\partial+1$
q-Dilation	Q	0	$qx\partial$
Cont. q-difference	Q	$Q-\mathrm{Id}$	$qx\partial+(q-1)x$
q-Differentiation	Q	$\frac{1}{(q-1)x}(Q-\mathrm{Id})$	$qx\partial+1$
q-Shift	Q	0	$qx\partial$
Discr. q-difference	Q	$Q-\mathrm{Id}$	$qx\partial+(q-1)x$
Eulerian operator	Id	$x\frac{d}{dx}$	$x\partial+x$
Mahlerian operator	M	0	$x^b\partial$
Divided differences	$f\mapsto f(a)$	$x\mapsto\frac{f(x)-f(a)}{x-a}$	$a\partial+1$

Table 2: Corresponding skew-polynomial rings and their commutation rules

or similar quantities, for example

$$\sum_{k=0}^{m}\binom{m}{k}B_{n+k}=(-1)^{m+n}\sum_{k=0}^{n}\binom{n}{k}B_{m+k}.$$

None of the quantities covered by these algorithms admits a definition via a ∂-finite ideal, but all three algorithms are based on principles that resemble those employed for holonomic systems and ∂-finite ideals. In each case, it turns out that the differential/difference equations defining the integrand/summand are of a form that permits to prove the existence of at least one non-trivial differential/difference equation for the integral/sum by means of a counting argument.

In this article, we give algorithms dealing with ideals of Ore algebras that are *not* ∂-finite. They generalize the algorithms known for the ∂-finite case and cover the extensions to non-holonomic functions discussed so far. Holonomy being lost, it is not always the case that creative telescoping can succeed. However, holonomy is only a sufficient condition. We show that by considering more generally the *dimension* of the ideals and another quantity that we call *polynomial growth*, it is possible to predict termination of a generalization of Chyzak's generalization of Zeilberger's fast algorithm. As special cases, we recover holonomic systems (dimension 0, polynomial growth 1), but also the special purpose algorithms mentioned above for Abel-type sums (dimension 2, polynomial growth 1), Stirling-number identities and summation identities about Bernoulli numbers (dimension 1, polynomial growth 1). In addition, we get for free a summation/integration algorithm that can deal with non-holonomic special functions such as the incomplete Gamma function $\Gamma(n,z)$, the Hurwitz zeta function $\zeta(n,z)$, polylogarithms $\mathrm{Li}_n(x)$, ... Examples are given in Section 4.4.

2. ORE ALGEBRAS AND THEIR IDEALS

Our motivation for using Ore algebras is the convenient polynomial representation of linear operators that they offer. Classical notions of commutative polynomial rings generalize to this setting. In this section, we recall without proof the basic definitions and facts we use (see [6, 8, 11] and their references for proofs, details, and history).

2.1 Ore Algebras

σ-derivations

Let A be a commutative algebra over a field k, and σ an injective algebra endomorphism of A that induces the identity on k. A k-linear endomorphism δ of A is called a *σ-derivation* if it satisfies the skew Leibniz rule $\delta(uv)=\sigma(u)\delta(v)+\delta(u)v$ for all u and v in A.

Skew-polynomial rings

The associative ring generated over A by a new indeterminate ∂ and the relations $\partial u=\sigma(u)\partial+\delta(u)$ for all $u\in A$ is called a *(left) skew-polynomial ring*. It is denoted by $A[\partial;\sigma,\delta]$. It does not have any zero-divisors. It is called an *extension* of the skew-polynomial ring $B[\partial;\sigma',\delta']$ when $B\subset A$, $\sigma'=\sigma|_B$ and $\delta'=\delta|_B$. In examples, we allow ourselves in the interest of increased readability to use different symbols in place of ∂. In particular, we will use S_n, S_k, etc. for denoting both the indeterminate ∂ and the corresponding σ in shift algebras.

Ore operators

A skew polynomial $L\in A[\partial;\sigma,\delta]$ acts on left $A[\partial;\sigma,\delta]$-modules—in most of our applications, these are modules of functions, power series, or sequences. Solving then means finding an element h in such a module such that $L\cdot h=0$. In this perspective, skew polynomials are called *Ore operators*.

For any $\lambda\in A$, the action $\partial: a\mapsto\partial\cdot a=\lambda\sigma(a)+\delta(a)$ turns the algebra A itself into a left $A[\partial;\sigma,\delta]$-module. For any u,v in A, one has the product rule $\partial\cdot uv=\sigma(u)\partial\cdot v+\delta(u)v$.

Tables 1 and 2 illustrate some common types of Ore operators when $A=k[x]$, together with the values of σ and δ that define the associated skew-polynomial ring.

In all these examples, σ and δ can be written as A-linear combinations of ∂ and the identity. We call *linear* a skew-polynomial ring with this property.

Ore algebras

Ore algebras are a generalization of skew-polynomial rings well suited to the manipulation of multivariate special functions. Let $C(\boldsymbol{x})=C(x_1,\dots,x_m)$ be a field of characteristic 0 of rational functions, $(\sigma_1,\dots,\sigma_n)$ be n C-algebra morphisms of $C(\boldsymbol{x})$ commuting pairwise, and for each i, let δ_i be a σ_i-derivation, such that the δ_i's commute pairwise and commute with the σ_j's when $j\neq i$. The *Ore algebra* $\mathbb{O}_{\boldsymbol{x}}=C(\boldsymbol{x})\langle\boldsymbol{\partial}\rangle$ is the associative $C(\boldsymbol{x})$-algebra generated by indeterminates $\boldsymbol{\partial}=\{\partial_1,\dots,\partial_n\}$ modulo the relations

$$\partial_i a=\sigma_i(a)\partial_i+\delta_i(a)\quad(a\in C(\boldsymbol{x})),\qquad \partial_i\partial_j=\partial_j\partial_i.\quad(1)$$

An Ore algebra $\mathbb{O}$ or an extension $A\otimes_{C(\boldsymbol{x})}\mathbb{O}$ of it is called *linear* when for each i, both σ_i and δ_i can be expressed as $C(\boldsymbol{x})$-linear combinations of ∂_i and the identity.

Given two Ore algebras $\mathbb{O}_{\boldsymbol{x}}$ and $\mathbb{O}_{\boldsymbol{t}}$, we write $\mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ or with an abuse of notation $C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x},\boldsymbol{\partial_t}\rangle$ for $\mathbb{O}_{\boldsymbol{x}} \otimes_C \mathbb{O}_{\boldsymbol{t}}$.

2.2 Ideals

We write $I \trianglelefteq R$ to denote that I is a left ideal in the ring R, and $I = \langle p_1, \ldots, p_k\rangle$ to denote that I is generated by $p_1, \ldots, p_k$. The study of a function or sequence f translates algebraically into the study of its *annihilating* ideal $\operatorname{ann}_A f$, i.e., the left ideal of operators in an appropriate algebra A that annihilate f. (We also write $\operatorname{ann} f$ when no ambiguity on A can arise.) Computations concern finding generators of this ideal or, at least, of a sufficiently large subideal of it. This is in particular the case for creative telescoping that we study here. It computes an ideal annihilating the definite integral or sum of interest starting from a description of an ideal annihilating the summand or integrand.

2.3 Gröbner Bases

Terms

If $f = \sum c_{\boldsymbol{\alpha}} \boldsymbol{\partial}^{\boldsymbol{\alpha}}$ is a polynomial, each $c_{\boldsymbol{\alpha}} \boldsymbol{\partial}^{\boldsymbol{\alpha}}$ for which $c_{\boldsymbol{\alpha}} \neq 0$ is called a *term* of f, $c_{\boldsymbol{\alpha}}$ is its coefficient, $\boldsymbol{\partial}^{\boldsymbol{\alpha}}$ its monomial, and $\boldsymbol{\alpha}$ its exponent. The total degree of f is the maximum $|\boldsymbol{\alpha}|$ over its terms, where we use the notation $|\boldsymbol{\alpha}| = |(\alpha_1, \ldots, \alpha_n)| = \alpha_1 + \cdots + \alpha_n$.

Monomial orders

A *monomial order* is a total order on the monomials that is compatible with the product and does not have infinite descending chains. A *graded order* is a monomial order such that $\boldsymbol{\partial}^{\boldsymbol{\alpha}} > \boldsymbol{\partial}^{\boldsymbol{\beta}}$ whenever $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$.

Gröbner bases

For a given monomial order, the leading term of a polynomial f is the term with largest monomial for that order. We write $\mathrm{E}(f)$ for its exponent in $\mathbb{N}^n$. The crucial property that lets the theory parallel that of the commutative case is that $\mathrm{E}(fg) = \mathrm{E}(f) + \mathrm{E}(g)$. The set $S_I := \{\mathrm{E}(f), f \in I\}$ thus has the property $S_I = S_I + \mathbb{N}^n$. The complement of S_I is a finite union of translates of coordinate subspaces. A *Gröbner basis* G of I is a set of generators I such that S_I is the sum of $\mathrm{E}(g) + \mathbb{N}^n$ over $g \in G$.

2.4 Hilbert Dimension

We write $R^{(s)}$ for the set of polynomials of an Ore algebra R of total degree at most s. Then $I^{(s)} := I \cap R^{(s)}$ is a vector space over $C(\boldsymbol{x})$. As in the commutative case, the Hilbert function of the ideal I is defined by $\mathrm{HF}_I(s) := \dim(R^{(s)}/I^{(s)}) = \dim R^{(s)} - \dim I^{(s)}$; for s large enough this function is equal to a polynomial whose degree is called the *(Hilbert) dimension* of the ideal. We denote this integer $\dim I$, or $\dim_R I$ when we want to make the ambient ring explicit. A reference for the results of this section is [10].

Example 1 (∂-finite ideals). *An ideal is* ∂-finite *if its dimension is 0. This special class of ideals has been studied a lot from the computational point of view: in this case, the quotient R/I is a finite-dimensional vector space, so that many techniques of linear algebra apply.*

Example 2 (Hypergeometric terms). *An n-variate sequence $u_{m_1,\ldots,m_n}$ is a* hypergeometric term *if*

$$\frac{u_{m_1,\ldots,m_{i-1},m_i+1,m_{i+1},\ldots,m_n}}{u_{m_1,\ldots,m_n}} \in \mathbb{Q}(m_1, \ldots, m_n)$$

for $i = 1, \ldots, n$. If $\boldsymbol{S} = (S_1, \ldots, S_n)$, S_i representing the shift operator*, the annihilating ideal of such a sequence in the algebra $\mathbb{Q}(\boldsymbol{m})\langle\boldsymbol{S}\rangle$ contains operators $S_1 - r_1(\boldsymbol{m}), \ldots, S_n - r_n(\boldsymbol{m})$. It is therefore ∂-finite and moreover, the dimension of the quotient as a vector space is 1.*

Example 3 (Stirling numbers). *Stirling numbers of the second kind, $S_2(n,k)$, satisfy the linear recurrence*

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k).$$

The ideal generated by this relation in $\mathbb{Q}(n,k)\langle S_n, S_k\rangle$ has dimension 1. Properties of the generating series imply that this is not a ∂-finite sequence, so that 1 is the lowest possible dimension to work with.

The dimension of an ideal can be computed from a Gröbner basis for a graded order: it is the largest dimension of coordinate subspaces of $\mathbb{N}^n$ that belong to the complement of the set of exponents of the leading terms [3, p. 449].

We note the following inequalities that also hold in this non-commutative context:

$$\begin{aligned} I \subset J &\Rightarrow \dim J \leq \dim I, \\ I \trianglelefteq C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x},\boldsymbol{\partial_t}\rangle &\Rightarrow \dim_{C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x}\rangle}(I \cap C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x}\rangle) \\ &\qquad \leq \dim_{C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x},\boldsymbol{\partial_t}\rangle} I. \end{aligned}$$

The first one follows from the inclusion of the vector spaces $I^{(s)} \subset J^{(s)}$. The second inequality involves dimensions relative to two different ambient rings. It can be seen by considering the following vector spaces: $F = C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x},\boldsymbol{\partial_t}\rangle^{(s)}$ contains $G = C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x}\rangle^{(s)}$ and $H = I^{(s)}$. Then the inequality follows from $G/(G \cap H) \subset F/H$.

We also make use of the following properties of dimension.

Lemma 1. *Let $I \trianglelefteq \mathbb{O}_{\boldsymbol{x}} = C(\boldsymbol{x})\langle\boldsymbol{\partial_x}\rangle$ with $\dim I = d$.*

1. *For any $\boldsymbol{\partial_t} \subset \boldsymbol{\partial_x}$, $|\boldsymbol{\partial_t}| \geq d+1 \Longrightarrow I \cap C(\boldsymbol{x})\langle\boldsymbol{\partial_t}\rangle \neq \{0\}$.*
2. *There exists $\boldsymbol{\partial_t} \subset \boldsymbol{\partial_x}$ of cardinality d such that $I \cap C(\boldsymbol{x})\langle\boldsymbol{\partial_t}\rangle = \{0\}$.*

Proof. We sketch the proof which shows how the notions presented so far interact (exactly as in the commutative case). By definition of the Hilbert function,

$$\begin{aligned} &\dim I^{(s)} + \dim C(\boldsymbol{x})\langle\boldsymbol{\partial_t}\rangle^{(s)} \\ &= \dim \mathbb{O}_{\boldsymbol{x}}^{(s)} + \dim C(\boldsymbol{x})\langle\boldsymbol{\partial_t}\rangle^{(s)} - \mathrm{HF}_I(s) \\ &= \dim \mathbb{O}_{\boldsymbol{x}}^{(s)} + \binom{|\boldsymbol{\partial_t}| + s}{s} - \mathrm{HF}_I(s). \end{aligned}$$

The binomial is a polynomial in s of degree $|\boldsymbol{\partial_t}| \geq d+1$ with positive leading coefficient, so that for large enough s, it is larger than $\mathrm{HF}_I(s)$. The sum of the dimensions of the vector subspaces on the left-hand side is therefore larger than the dimension of $\mathbb{O}_{\boldsymbol{x}}^{(s)}$ and thus they intersect nontrivially.

The second part follows by considering the exponents of leading terms. Now the combinatorial theory is exactly as in [7, Ch. 9]. There exists a coordinate subspace of $\mathbb{N}^n$ of dimension d in the complement set. This means that there exists a subset $\boldsymbol{\partial_t} \subset \boldsymbol{\partial_x}$ such that no monomial in $\boldsymbol{\partial_t}$ is a leading term of an element of I. Thus there cannot be an element of I in those variables only, as was to be proved. $\square$

3. CLOSURE PROPERTIES

Our main result, Thm. 3 in the next section, generalizes the fact that holonomy is preserved under definite integration. First, we show how addition, multiplication, and action by ∂ behave with respect to dimension, generalizing the corresponding closure properties for ∂-finite ideals.

THEOREM 1 (CLOSURE PROPERTIES). *Let $I_1, I_2 \trianglelefteq \mathbb{O}_{\boldsymbol{x}}$ and let f_1, f_2 be annihilated by I_1, I_2, respectively. Then:*

1. $\dim \operatorname{ann}(\partial \cdot f_1) \leq \dim I_1$ *for all* ∂ *in* $\{\boldsymbol{\partial_x}\}$.
2. $\dim \operatorname{ann}(f_1 + f_2) \leq \max(\dim I_1, \dim I_2)$.
3. *If f_1, f_2 belong to the coefficient ring of a linear extension of $\mathbb{O}_{\boldsymbol{x}}$, then* $\dim \operatorname{ann}(f_1 f_2) \leq \dim I_1 + \dim I_2$.

PROOF. We show part 3. The arguments for parts 1 and 2 are similar and simpler.

Setting $k := \dim I_1 + \dim I_2 + 1$, it suffices to show that any k elements $\partial_1, \ldots, \partial_k$ among $\boldsymbol{\partial}$ are dependent modulo $\operatorname{ann}(f_1 f_2)$.

Given a Gröbner basis for I_1 with respect to a graded order, each polynomial $P \in \mathbb{O}_{\boldsymbol{x}}$ can be reduced to a normal form $\overline{P}$ such that $P \cdot f_1 = \overline{P} \cdot f_1$ and $\deg \overline{P} \leq \deg P$. Moreover, by definition of the dimension, the set of all monomials in $\{\overline{P} \mid P \in R^{(s)}\}$ has cardinality growing like $O(s^{\dim I_1})$. The same considerations hold for f_2.

By induction on the degree of P, the condition of a linear extension implies that $P \cdot (f_1 f_2)$ rewrites as a linear combination of monomials $(\boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot f_1)(\boldsymbol{\partial}^{\boldsymbol{\gamma}} \cdot f_2)$, with $|\boldsymbol{\beta} + \boldsymbol{\gamma}| \leq \deg P$. Moreover, we can assume that the monomials have been reduced to their normal forms and the inequalities still hold.

Let $s \geq 0$ and consider the following identities

$$\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k} \cdot f_1 f_2 = \sum_{|\boldsymbol{\beta}| \leq s} \sum_{|\boldsymbol{\gamma}| \leq s} u_{\boldsymbol{\alpha};\boldsymbol{\beta},\boldsymbol{\gamma}}(\boldsymbol{x}) \big(\boldsymbol{\partial}^{\boldsymbol{\beta}} \cdot f_1\big)\big(\boldsymbol{\partial}^{\boldsymbol{\gamma}} \cdot f_2\big), \tag{2}$$

($|\boldsymbol{\alpha}| \leq s$), where the sums on the right are constructed as above. Then the first sum actually ranges over a subset $\{\boldsymbol{\beta}, |\boldsymbol{\beta}| \leq s\}$ of cardinality $O(s^{\dim I_1})$ and the second one ranges over a subset $\{\boldsymbol{\gamma}, |\boldsymbol{\gamma}| \leq s\}$ of cardinality $O(s^{\dim I_2})$. Thus there is a generating set of $O(s^{\dim I_1} s^{\dim I_2}) = O(s^{k-1})$ monomials for all the summands. This implies that for s large enough, there exists a nontrivial linear combination of the $O(s^k)$ polynomials in (2) of the form

$$\sum_{|\boldsymbol{\alpha}| \leq s} w_{\boldsymbol{\alpha}}(\boldsymbol{x}) \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k} \cdot (f_1 f_2) = 0,$$

as we wanted to show. □

Algorithm

The proof gives an algorithm that computes generators of a subideal of the desired annihilating ideal, with a dimension that obeys the inequality. For increasing s, compute the normal forms of all monomials in $\boldsymbol{\partial}$ of degree at most s, compute linear combinations between them (the kernel of the matrix $(u_{\boldsymbol{\alpha};\boldsymbol{\beta},\boldsymbol{\gamma}})$), and return these relations if they are sufficiently many to obtain the dimension of the theorem. For ∂-finite ideals, this returns the same result as the algorithms in [6]. Various optimizations are possible. The other closure operations are similar.

EXAMPLE 4. *The sequence*

$$f_{n,m,k,l} = \binom{n}{k} S_2(k,l) S_2(n-k,m),$$

is annihilated by an ideal of dimension at most 2. This follows from Thm. 1 by observing: (i) *that $\binom{n}{k}$ is hypergeometric, thus ∂-finite, and thus annihilated by an ideal of dimension 0;* (ii) *that $S_2(k,l)$ and $S_2(n-k,m)$ are Stirling-like (see Section 4.4) and thus annihilated by certain ideals of dimension 1. More specifically, the factors $\binom{n}{k}$, $S_2(k,l)$, and $S_2(n-k,m)$ are annihilated by the ideals*

$$\langle (k-n-1)S_n + n + 1, (k+1)S_k + k - n, S_m - 1, S_l - 1 \rangle,$$
$$\langle S_n - 1, S_k S_l - (l+1)S_l - 1, S_m - 1 \rangle,$$
$$\langle S_n S_k - 1, (m+1)S_m S_k + S_k - S_m, S_l - 1 \rangle,$$

respectively. The algorithm sketched above yields

$$\begin{aligned} I := \langle & 1 + n + (1+m)(1+n)S_m - (1-k+n)S_n S_m, \\ & (k-n)S_m + (1+k)S_k S_l + (1+k)(1+m)S_k S_l S_m \\ & + (1+l)(k-n)S_l S_m, 1 + n + (1+l)(1+n)S_l \\ & - (1+k)S_k S_l S_n \rangle \trianglelefteq C(n,m,k,l)\langle S_n, S_m, S_k, S_l \rangle \end{aligned}$$

as an ideal of annihilators of $f_{n,m,k,l}$. It has dimension 2.

4. CREATIVE TELESCOPING

Creative telescoping is basically a combination of differentiation under the integral sign and integration by parts, or analogues for other operators. We now give it an algebraic interpretation. Our main theorem can be viewed as predicting cases when identities are bound to exist.

4.1 Telescoping of an Ideal

The heart of the method of creative telescoping translates algebraically into the notion of the telescoping of an ideal.

DEFINITION 1. *Let $I \trianglelefteq \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}} = C(\boldsymbol{x},\boldsymbol{t})\langle \boldsymbol{\partial_x}, \boldsymbol{\partial_t} \rangle$ be a left ideal. Assume $|\boldsymbol{t}| = |\boldsymbol{\partial_t}|$. We define the* telescoping *of I with respect to $\boldsymbol{t} = (t_1, \ldots, t_k)$ as the left ideal of $C(\boldsymbol{x})\langle \boldsymbol{\partial_x} \rangle$*

$$T_{\boldsymbol{t}}(I) := (I + \partial_{t_1} \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}} + \cdots + \partial_{t_k} \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}) \cap C(\boldsymbol{x})\langle \boldsymbol{\partial_x} \rangle.$$

DEFINITION 2. *The variables $\boldsymbol{\partial_t} = (\partial_{t_1}, \ldots, \partial_{t_k})$ of the Ore algebra $\mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ are* telescopable *if there exist elements $a_1, \ldots, a_k$ in $C(\boldsymbol{x},\boldsymbol{t})$ such that*

$$\delta_{t_i}(a_i) \in C(\boldsymbol{x}) \setminus \{0\} \quad \text{and} \quad \sigma_{t_i}(a_i)\partial_{t_j} = \partial_{t_j}\sigma_{t_i}(a_i) \quad (j \neq i).$$

Note that this is a condition on the algebra, and does not depend on any specific ideal. In view of Table 1, this is not a strong restriction for our applications. For example, the differential operator d/dt and the difference operator Δ_t are telescopable, with $a = t$, but the shift operator is not. This notion lets us generalize an idea of Wegschaider [14] used at the end of the proof of Thm. 3 below.

4.2 Polynomial Growth

In Thm. 3 below we give an upper bound for the dimension of $T_{\boldsymbol{t}}(I)$, thus providing a termination criterion for the algorithms in Section 5. Our bound depends on the dimension of I and on its "polynomial growth", defined as follows.

DEFINITION 3. *The left ideal $I \trianglelefteq \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ has* polynomial growth *p with respect to a given graded order if there exists a sequence of polynomials $P_s(\boldsymbol{x},\boldsymbol{t})$, $s \in \mathbb{N}$, such that for*

any $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le s$, the normal form of $P_s(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\partial}^\alpha$ with respect to a Gröbner basis of I for the order has coefficients in $C(\boldsymbol{x})[\boldsymbol{t}]$ whose degrees with respect to $\boldsymbol{t}$ are $O(s^p)$. We say that I has polynomial growth p when there is a graded order with respect to which it does.

If all the σ's are automorphisms, the polynomial growth is bounded by the dimension of the Ore algebra, $|\boldsymbol{x}|+|\boldsymbol{t}|$. But the interesting cases are those where the polynomial growth is smaller than that. For certain ideals of dimension 0, we get a better estimate in Thm. 2 below. For an arbitrary ideal in an arbitrary Ore algebra, we do not know how to determine its polynomial growth algorithmically yet.

Example 5. *The basis at the end of Example 4 is a Gröbner basis with respect to a graded order. Inspection of its leading coefficients shows that the sequence*

$$P_s(n,m,k) := \prod_{|j|\le s} (1+n-k+j)(1+k+j)^2$$

satisfies the conditions of Definition 3. Since $\deg P_s = O(s)$, *it follows that I has polynomial growth 1.*

Just before stating our result, we now give notation and a definition for algebras amenable to it.

Definition 4. *Let R be $C(\boldsymbol{x})[\boldsymbol{t}]$ and $R_{\le n}$ the set of elements $A \in R$ such that $\deg_{\boldsymbol{t}} A \le n$. A* difference-differential algebra *is an Ore algebra $\mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ such that for any i, either*

- *$\delta_i = 0$ and, for any $u \in R$, $\deg_{\boldsymbol{t}} \sigma_i(u) = \deg_{\boldsymbol{t}} u$;*
- *or $\sigma_i = \mathrm{Id}$ and δ_i is a derivation such that, for any $u \in R$, $\deg_{\boldsymbol{t}} \delta_i(u) \le \nu + \deg_{\boldsymbol{t}} u$,*

$\nu \in \mathbb{N}$ being fixed. We set $\mathcal{S}$ and $\mathcal{D}$ to be the sets of i of, resp., first and second types.

Theorem 2. *Let $\mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ be a difference-differential algebra endowed with a graded ordering. Let $I \trianglelefteq \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ have dimension 0. Call ϕ the map sending any $A \in \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ to its normal form modulo I w.r.t. the given graded ordering, and Γ the (finite) set of monomials in normal form. Then, there exist $m,\ell \in \mathbb{N}$ and $L \in R_{\le \ell}$ such that, for any i and any $\boldsymbol{\beta} \in \Gamma$,*

$$\phi(\partial_i \boldsymbol{\partial}^{\boldsymbol{\beta}}) \in \frac{1}{L}\sum_{\gamma\in\Gamma} R_{\le m}\boldsymbol{\partial}^{\gamma}. \tag{3}$$

Define a sequence $(P_s)_{s\in\mathbb{N}}$ by $P_0 = 1$ and

$$P_{s+1} := \begin{cases} \operatorname{lcm}(P_s, \operatorname{lcm}(L\sigma_i(P_s)), P_s \operatorname{lcm}(L, Q_s)), & \text{if } \mathcal{D} \ne \emptyset, \\ \operatorname{lcm}(P_s, \operatorname{lcm}(L\sigma_i(P_s))), & \text{if } \mathcal{D} = \emptyset, \end{cases}$$

where Q_s denotes the squarefree part of P_s and the lcm in σ_i is over all i.

If $\deg_{\boldsymbol{t}} P_s = \Theta(s^p)$ for some integer $p > 0$, then I has polynomial growth p.

Before the proof, note that, by the definition of ϕ, the sum in (3) is limited to $|\gamma| \le |\boldsymbol{\beta}|+1$. Moreover, the existence of a uniform (L,m) is easily obtained in the zero-dimensional case, as the set of $(i,\boldsymbol{\beta})$ is then finite.

Proof. Introduce the sets $F_s = \{\boldsymbol{\partial}^{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| \le s\}$ for $s \in \mathbb{N}$. For $n \in \mathbb{N}$, define $R_{\le n}[\Gamma]$ as $\sum_{\gamma\in\Gamma} R_{\le n}\boldsymbol{\partial}^{\gamma}$. For a fixed s, suppose there exists $P \in R$ and an integer D for which

$$\phi(F_s) \subset P^{-1}R_{\le D}[\Gamma].$$

We look for homologues P' and D' for F_{s+1}. As ϕ satisfies

$$\phi(F_{s+1}) = \phi\Big(F_s \cup \bigcup_i \partial_i F_s\Big) \subset \phi(F_s) \cup \bigcup_i \phi\big(\partial_i \phi(F_s)\big),$$

we study $\phi(\partial_i A)$ for $A = P^{-1}U\boldsymbol{\partial}^{\boldsymbol{\beta}}$ when $U \in R_{\le D}$, $\boldsymbol{\beta} \in \Gamma$, and $|\boldsymbol{\beta}| \le s$. If $i \in \mathcal{S}$, then from $\delta_i = 0$ follows $\partial_i A = \sigma_i(P)^{-1}\sigma_i(U)\partial^i\boldsymbol{\partial}^{\boldsymbol{\beta}}$; therefore,

$$\phi(\partial_i A) \in \big(L\sigma_i(P)\big)^{-1} R_{\le D+m}[\Gamma].$$

Else, $i \in \mathcal{D}$ and from σ_i being the identity follows $\partial_i A = P^{-1}U\partial^i\boldsymbol{\partial}^{\boldsymbol{\beta}} + P^{-1}\delta_i(U)\boldsymbol{\partial}^{\boldsymbol{\beta}} + \delta_i(P^{-1})U\boldsymbol{\partial}^{\boldsymbol{\beta}}$; thus, $\phi(\partial_i A)$ is in

$$\frac{1}{LP}R_{\le D+m}[\Gamma] + \frac{1}{LP}R_{\le D+\ell+\nu}\boldsymbol{\partial}^{\boldsymbol{\beta}} + \frac{1}{PQ}R_{\le D+D_0}\boldsymbol{\partial}^{\boldsymbol{\beta}}$$

where Q is the square-free part of P and $D_0 = \deg_{\boldsymbol{t}} Q$. Defining P' as the lcm of P, $P\operatorname{lcm}(L,Q)$ if $\mathcal{D} \ne \emptyset$, and the $\sigma_i(P)L$'s for $i \in \mathcal{S}$ yields $A, \phi(\partial_i A) \in (P')^{-1}R[\Gamma]$. Next, setting $D_1 = \deg_{\boldsymbol{t}}(P'/PQ)$ and $D_2 = \deg_{\boldsymbol{t}}(P'/LP)$, then Δ as the maximum of $\max\{m,\ell\} + D_2$ and $\max\{\ell+\nu+D_2, D_0+D_1\}$ if $\mathcal{D} \ne \emptyset$, finally $D' = D + \Delta$ yields

$$\phi(F_{s+1}) \subset (P')^{-1}R_{\le D'}[\Gamma].$$

Therefore, introducing the sequence (P_s) defined in the theorem, we get by a first induction on s that $\phi(F_s) \subset P_s^{-1}R[\Gamma]$. Next, under the additional assumption that $\deg_{\boldsymbol{t}} P_s = \Theta(s^p)$ for some integer $p > 0$, P and P' in the proof above can be taken as P_s and P_{s+1}, so that all of D_0, D_1, and D_2 are bounded by $\deg P_{s+1} - \deg P_s$, thus by $O(s^{p-1})$. Therefore, by another induction

$$\phi(F_s) \subset P_s^{-1}R_{\le O(s^p)}[\Gamma],$$

proving a polynomial growth p for I. □

In the uses of creative telescoping for summation, the identities (or ideals) are often stated in terms of shifts, while the operation of interest is a finite difference. However, it is possible to compute the polynomial growth in a difference-differential algebra with shifts, and it will be the same when considered in an algebra with difference operators. This is the meaning of the following theorem, whose proof based on transporting graded orderings we omit.

Proposition 1. *For a given difference-differential algebra $\mathbb{O}_{\boldsymbol{x},\boldsymbol{t}}$ and indeterminates $\boldsymbol{u} = (u_1,\dots,u_k)$, consider*

$$\mathbb{O}_{\boldsymbol{x},(\boldsymbol{t},\boldsymbol{u})} = \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}} \otimes_C C(\boldsymbol{u})[\partial_{u_1}; S_{u_1}, 0]\dots[\partial_{u_k}; S_{u_k}, 0] \quad \text{and}$$
$$\mathbb{O}'_{\boldsymbol{x},(\boldsymbol{t},\boldsymbol{u})} = \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}} \otimes_C C(\boldsymbol{u})[\partial_{u_1}; S_{u_1}, \Delta_{u_i}]\dots[\partial_{u_k}; S_{u_k}, \Delta_{u_k}],$$

where $\Delta_{u_i} = S_{u_i} - \mathrm{Id}$ $(i = 1,\dots,k)$. Let μ be the (bijective) left-$C(\boldsymbol{x},\boldsymbol{t},\boldsymbol{u})$-linear map sending each ∂_{u_i} to $\partial_{u_i}+1$ and all the $\boldsymbol{\partial}_{\boldsymbol{x}}, \boldsymbol{\partial}_{\boldsymbol{t}}$ to themselves. Then an ideal $I \trianglelefteq \mathbb{O}_{\boldsymbol{x},(\boldsymbol{t},\boldsymbol{u})}$ and $\mu(I) \trianglelefteq \mathbb{O}'_{\boldsymbol{x},(\boldsymbol{t},\boldsymbol{u})}$ have the same polynomial growth.

4.3 Main Result

Our main result is the following sufficient condition for creative telescoping. For simplicity, we state it for $|\boldsymbol{t}| = |\boldsymbol{\partial}_{\boldsymbol{t}}|$.

Theorem 3 (Creative Telescoping). *Let $I \trianglelefteq \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}} = C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial}_{\boldsymbol{x}},\boldsymbol{\partial}_{\boldsymbol{t}}\rangle$ be an ideal of dimension d and polynomial growth p. If $|\boldsymbol{t}| = |\boldsymbol{\partial}_{\boldsymbol{t}}|$ and the $\boldsymbol{\partial}_{\boldsymbol{t}}$'s are telescopable then*

$$\dim_{\mathbb{O}_{\boldsymbol{x}}} T_{\boldsymbol{t}}(I) \le d + (p-1)|\boldsymbol{t}|,$$

whenever this bound is nonnegative. In particular, when the bound is smaller than $|\boldsymbol{x}|$, the ideal $T_{\boldsymbol{t}}(I)$ is non-trivial.

Proof. If $d+(p-1)|\boldsymbol{t}| \geq |\boldsymbol{x}|$, there is nothing to show. Otherwise, by Lemma 1, it is sufficient to show that any $k := d+(p-1)|\boldsymbol{t}|+1$ elements $\partial_1, \ldots, \partial_k$ in $\{\boldsymbol{\partial_x}\}$ are dependent modulo $T_{\boldsymbol{t}}(I)$.

Let $s \geq 0$ and consider the following set of members of I from the definition of the polynomial growth of I:

$$P_s(\boldsymbol{x},\boldsymbol{t})\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k}\boldsymbol{\partial_t}^{\boldsymbol{\alpha_t}} - \sum_{|\boldsymbol{\beta}|\leq s} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\partial}^{\boldsymbol{\beta}}, \quad |\boldsymbol{\alpha}| \leq s. \quad (4)$$

The coefficients $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x},\boldsymbol{t})$ can be viewed as $C(\boldsymbol{x})$-linear combinations of monomials in $\boldsymbol{t}$ of degree $O(s^p)$. The ideal I having dimension d, these sums in (4) for all $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| \leq s$ actually range over a common subset of $\{\boldsymbol{\beta}, |\boldsymbol{\beta}| \leq s\}$ of cardinality $O(s^d)$. Thus, there is a generating set of $O(s^{d+p|\boldsymbol{t}|})$ monomials in $\boldsymbol{t}, \boldsymbol{\partial}$ for all the summands. This implies that for s large enough, there exists a nontrivial linear combination of the $O(s^{k+|\boldsymbol{t}|})$ polynomials in (4) of the form

$$P_s(\boldsymbol{x},\boldsymbol{t}) \sum_{|\boldsymbol{\alpha}|\leq s} C_{\boldsymbol{\alpha}}(\boldsymbol{x})\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k}\boldsymbol{\partial_t}^{\boldsymbol{\alpha_t}} =: P(s)Q \in I.$$

Multiplying by $1/P_s(\boldsymbol{x},\boldsymbol{t})$ shows that $Q \in I \cap C(\boldsymbol{x})\langle \boldsymbol{\partial_x}, \boldsymbol{\partial_t}\rangle$. Now the operators ∂_{t_i}'s commute with the coefficients of Q. Thus, Q can be rewritten

$$Q = R + \partial_{t_1}Q_1 + \cdots + \partial_{t_{|t|}}Q_{|t|}, \quad (5)$$

with $R \in C(\boldsymbol{x})\langle \partial_1, \ldots, \partial_k\rangle \in T_{\boldsymbol{t}}(I)$. If $R \neq 0$, we have found the element of $T_{\boldsymbol{t}}(I)$ we were looking for. Otherwise, since $Q \neq 0$, there exists a $Q_i \neq 0$. Since ∂_{t_i} is telescopable, there is an element $a_i \in C(\boldsymbol{x},\boldsymbol{t})$ such that

$$\sigma_{t_i}(a_i)\partial_{t_i} = b_i + \partial_{t_i}a_i, \qquad \sigma_{t_i}(a_i)\partial_{t_j} = \partial_{t_j}\sigma_{t_i}(a_i)$$

for some $b_i \in C(x) \setminus \{0\}$. Multiplying (5) by $\sigma_{t_i}(a_i)$ yields

$$I \ni \sigma_{t_i}(a_i)Q = b_iQ_i + \partial_{t_i}\tilde{Q}_i + \cdots + \partial_{t_{|t|}}\tilde{Q}_{|t|}.$$

Now, $Q_i \neq 0$ can be rewritten as in (5). Repeating this process if necessary, we eventually reach a non-zero element of $T_{\boldsymbol{t}}(I)$, as was to be proved. □

4.4 Examples

Proper hypergeometric sequences

These are sequences with two indices that are hypergeometric as in Example 2 with the further constraint that they can be written

$$u_{m,k} = Q(m,k)\frac{\prod_{i=1}^{u}(a_im+b_ik+c_i)!}{\prod_{i=1}^{v}(u_im+v_ik+w_i)!}\xi^k, \quad (6)$$

where $\xi \in \mathbb{C}$, Q is a polynomial and the a_i's, b_i's, u_i's, v_i's are integers. A typical example is the binomial coefficient $\binom{m}{k}$. Since such a sequence is hypergeometric, a Gröbner basis of $\operatorname{ann} u_{m,k}$ for any order is formed by the relations

$$S_m - u_{m+1,k}/u_{m,k} \quad \text{and} \quad S_k - u_{m,k+1}/u_{m,k}.$$

The normal form of $S_m^{s_1}S_k^{s_2}$ with respect to this basis is simply the rational function $u_{m+s_1,k+s_2}/u_{m,k}$. Since the a_i, b_i, u_i, v_i's are all integers, the common denominator of all these rational functions for $s_1+s_2 \leq s$ has a degree that grows only *linearly* with s [13], as it is bounded by

$$P_s(m,k) = Q(m,k)\prod_{|j|\leq(|u_i|+|v_i|)s}\prod_i(u_im+v_ik+w_i+j) \times \prod_{|j|\leq(|a_i|+|b_i|)s}\prod_i(a_im+b_ik+c_i+j).$$

In our terminology, the corresponding annihilating ideal has polynomial growth 1, by Thm. 2 and Prop. 1. Further generalizations to the multivariate and q-cases can also be considered. Thm. 3 (with $d=0, p=1, |\boldsymbol{t}|=1, n=2$) generalizes the result that creative telescoping applies to proper hypergeometric sequences.

General hypergeometric sequences

Not every hypergeometric sequence is proper. For example,

$$u_{m,k} = \frac{1}{mk+1}\binom{2m-2k-1}{m-1}$$

is not. By a criterion of Abramov [1], creative telescoping fails on this example, i.e., $T_{\boldsymbol{t}}(I) = \{0\}$. This phenomenon is well consistent with our theorem, because the nonlinear factor in the denominator implies that the annihilating ideal I has polynomial growth 2: a possible choice for P_s is

$$P_s(m,k) = \prod_{i+j\leq s}((m+i)(k+j)+1) \times \prod_{|i|\leq 3s}(m-2k+i),$$

whose degree is quadratic in s, hence the polynomial growth by Thm. 2 and Prop. 1. Thm. 3 (with $d=0, p=2, |\boldsymbol{t}|=1, n=2$) implies the trivial bound $\dim T_{\boldsymbol{t}}(I) \leq 1$, which is reached in this example.

Holonomic Functions

This notion was popularized for special functions in [16]. The technical definition is related to holonomic D-modules. It turns out to be sufficient to consider ∂-finite ideals.

Proposition 2. *Let $I \trianglelefteq \mathbb{O}_{\boldsymbol{x},\boldsymbol{t}} = C(\boldsymbol{x},\boldsymbol{t})\langle \boldsymbol{\partial}_{\boldsymbol{x},\boldsymbol{t}}\rangle$ be a left ideal of dimension 0 of the* differential *Ore algebra ($\sigma_i = \mathrm{Id}$, $\delta_i = d/dx_i$). Then I has polynomial growth 1.*

Proof. The crucial property is that in the differential setting, the leading coefficient of an element does not change upon multiplication by some ∂_i (see Table 2). The result is a corollary of Thm. 2, which is applicable since I has dimension 0. The polynomials P_s defined in the theorem here take the form $P_s = L^s$, having degree $O(s^1)$ and thereby proving a polynomial growth 1. □

Put together, Prop. 2 and Thm. 3 imply $\dim T_{\boldsymbol{t}}(I) = 0$, and thus recover the celebrated closure of holonomic functions under definite integration as a special case.

Stirling-like sequences

This notion was introduced in [9]. A sequence is called Stirling-like if it is annihilated by an ideal of operators

$$\langle u + vS_k^{\alpha}S_{m_1}^{\beta} + wS_k^{\gamma}S_{m_1}^{\delta}, s_2 + t_2S_{m_2}, \ldots, s_n + t_nS_{m_n}\rangle,$$

where $u, v, w, s_2, \ldots, s_n, t_2, \ldots, t_n$ are polynomials that split into integer-linear factors, and $\alpha, \beta, \gamma, \delta$ are integers subject to a certain nondegeneracy condition. A typical example is $\binom{m}{k}S_2(k,m)$, where S_2 refers to the Stirling numbers of the second kind. It was shown in [9] that such ideals lead

to non-trivial creative telescoping relations whenever $n \geq 2$. These ideals have dimension 1 and polynomial growth 1, so our theorem (with $d = 1, p = 1, |\boldsymbol{t}| = 1, n \geq 3$) includes this result as a special case.

Abel-type sequences

This notion was introduced in [12]. A sequence is called Abel-type if it can be written in the form

$$u_{m,k}(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$$

for some proper hypergeometric term $u_{m,k}$. The annihilating ideals of such sequences can be written

$$\langle aS_mS_k - bS_r, cS_m - dS_s\rangle$$

for certain polynomials a, b, c, d. It was shown in [12] that such ideals lead to non-trivial creative telescoping relations. These ideals have dimension 2, and if $u_{m,k}$ is as in (6), then

$$P_s(m,r,s,k) = \tilde{P}_s(m,k) \times \prod_{|j|\leq s}(k+r+j)(m-k+j),$$

with $\tilde{P}_s(m,k)$ being the polynomial sequence stated above for proper hypergeometric terms, justifies that their polynomial growth is 1, so our theorem (with $d = 2$, $p = 1$, $|\boldsymbol{t}| = 1$, $n = 4$) includes this result as a special case, too.

Bernoulli examples

Chen and Sun [4] do not give a formal description of the class of summands to which their algorithm is applicable. Most of their examples concern sums with summands of the form $h_{k,m_1,m_2}B_{ak+bm_1+cm_2}$ where h is a hypergeometric term, B refers to Bernoulli numbers, Bernoulli polynomials, Euler numbers, or Euler polynomials, and a, b, c are specific integers. The annihilating ideals of such objects have dimension 1, and, if h is proper hypergeometric, polynomial growth 1. In this case, our theorem guarantees the success of creative telescoping.

Further examples

Our theorem also extends to sequences and functions for which no special purpose summation or integration algorithm has been formulated so far. For instance, the sequence $f_{n,m,k,l}$ considered in Examples 4 and 5 is not Stirling-like, but only annihilated by an ideal of dimension 2 and polynomial growth 1. Our Theorem predicts the existence of creative telescoping relations (cf. also Examples 6 and 7 below). We list some further identities from the literature that were previously considered inaccessible to computer algebra, but that can be proven with creative telescoping. In all the following identities, the integrand is not holonomic, but annihilated by an ideal of dimension 1 and polynomial growth 1, so our theorem predicts a priori that relations for the integral must exist.

$$\int_0^\infty x^{k-1}\zeta(n,\alpha+\beta x)\,dx = \beta^{-k}B(k,n-k)\zeta(n-k,\alpha),$$

$$\int_0^\infty x^{\alpha-1}\operatorname{Li}_n(-xy)\,dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)},$$

$$\int_0^\infty x^{k-1}\exp(xy)\Gamma(n,xy)\,dx = \frac{\pi y^{-k}}{\sin((n+k)\pi)}\frac{\Gamma(k)}{\Gamma(1-n)},$$

$$\int_0^\infty \frac{x}{(x^2+y^2)\sin(xz)}\,dx = \frac{\pi}{2\sinh(yz)}.$$

In these identities, B refers to the Beta-function, ζ to the Hurwitz zeta function, Li_n refers to the nth polylogarithm, and Γ to the incomplete Gamma function. Greek letters refer to parameters, n, m, k are discrete variables, x, y, z are continuous ones. Note that the polylogarithm, while holonomic for each specific n, is not even ∂-finite when n is "symbolic". Note also that the integrand of the last integral, despite being elementary, is not holonomic.

All these identities are proven by computing operators annihilating both sides, by making use of closure properties for sums, products or ∂ (Section 3) and by creative telescoping in the case of definite sums and integrals (Section 5 below). If the operators found in this way generate an ideal of dimension 0, then proving the identity reduces to verifying a finite number of initial values. For higher dimensional ideals, the number of initial values to be checked may be infinite. For example, for completing the proof of the first integral identity above, it remains to check the identity for $k = 1$ and *all* $n \geq 0$:

$$\int_0^\infty \zeta(n,\alpha+\beta x)dx = \frac{1}{\beta(n-1)}\zeta(n-1,\alpha),$$

an identity not much easier than the original one. In many instances, however, the identities to be verified as initial conditions are trivial.

5. ALGORITHMS FOR SUMMATION AND INTEGRATION

5.1 A Fasenmyer-Style Algorithm

The first algorithmic approach to symbolic summation goes back to Fasenmyer and was formulated for the summation of hypergeometric terms [13]. The following generalized version of her algorithm is applicable to any given annihilating ideal in an Ore algebra.

The algorithm follows from the proof of Thm. 3. Given an ideal $I \trianglelefteq C(\boldsymbol{x},\boldsymbol{t})\langle\boldsymbol{\partial_x},\boldsymbol{\partial_t}\rangle$, the algorithm searches for elements of $I \cap C(\boldsymbol{x})\langle\boldsymbol{\partial_x},\boldsymbol{\partial_t}\rangle$ (called $\boldsymbol{t}$-free operators). Proceeding by increasing total degree, it makes an ansatz with undetermined coefficients for such an operator, reduces it to normal form with respect to a Gröbner basis for I, brings the normal form to a common denominator, and then compares the coefficients with respect to the $\boldsymbol{t}$'s of the numerator to zero. This gives a linear system of equations over $C(\boldsymbol{x})$ for the undetermined coefficients, which is then solved. If the system has no solution, the procedure is repeated with a larger total degree. If a solution is found, it leads to an operator that can be written in the form (5)

$$A + \partial_{t_1}B_1 + \cdots + \partial_{t_{|t|}}B_{|t|} \in I$$

where $0 \neq A \in T_{\boldsymbol{t}}(I)$, by following the steps at the end of the proof that ensure $A \neq 0$.

The procedure can be repeated to search for further operators in $T_{\boldsymbol{t}}(I)$ until all operators found generate an ideal of the dimension predicted by Thm. 3.

Example 6. *For the ideal I from Example 4, with k as summation variable, this algorithm discovers a first operator for total degree 4 that may be written in the form*

$$S_m + S_l + (2+l+m)S_lS_m - S_lS_mS_n + (S_k-1)((m+1)S_mS_l - S_mS_nS_l + S_l).$$

As a consequence, we find

$$A := S_m + S_l + (2 + l + m)S_l S_m - S_l S_m S_n \in T_k(I).$$

Therefore A belongs to the annihilating ideal of the sum

$$\sum_k \binom{n}{k} S_2(k, l) S_2(n - k, m).$$

This proves that the sum is equal to $\binom{l+m}{l} S_2(n, l + m)$: A also belongs to the annihilating ideal of that quantity, and it is easily seen to agree with the sum for $n = 0$ and arbitrary l and m.

5.2 A Zeilberger-Style Algorithm

Zeilberger's "fast algorithm" was originally formulated for summation of proper hypergeometric terms only [15], then a differential analog was given for hyperexponential functions by Almkvist and Zeilberger [2]. This was later extended by Chyzak [5] to an integration/summation algorithm for arbitrary ∂-finite functions, with termination guaranteed in the holonomic case. We extend this algorithm further to the case of integrands or summands defined by arbitrary annihilating ideals.

The fast approach is applicable only for single sums or integrals, i.e., if $|\boldsymbol{t}| = 1$. (Refer however to [5, Sec. 3.3] for an iterated treatment of summations and integrations.) Let $I \trianglelefteq C(\boldsymbol{x}, t)\langle \boldsymbol{\partial_x}, \partial_t \rangle$ be given by a Gröbner basis G with respect to a graded order. The algorithm searches for operators

$$A + \partial_t B \in I$$

with $A \in C(\boldsymbol{x})\langle \boldsymbol{\partial_x} \rangle$ and $B \in C(\boldsymbol{x}, t)\langle \boldsymbol{\partial_x}, \partial_t \rangle$. Proceeding by increasing total degree, it makes an ansatz for both A and B and computes the normal form of $Q := A + \partial_t B$ with respect to G. Without loss of generality, only irreducible terms need to be included in the ansatz for B. In order for Q to belong to I, it is necessary and sufficient that all the coefficients in the normal form of Q be zero. Comparing them to zero leads to a system of first order linear functional equations (a "coupled system"), which is then solved. While in the ∂-finite case this system is always square, in the case of positive dimension it may be rectangular, owing to extraneous equations potentially being introduced by irreducible terms in $\partial_t B$ of degree $d + 1$. However, it is always possible to separate the equations into a square coupled system (that can be solved in a first step as in Chyzak's algorithm) and a system with additional linear algebraic constraints (that can be accommodated in a second step by computing suitable linear combinations of the solutions obtained in the first step). Any solution $A + \partial_t B \in I$ found in this way gives rise to an element $A \in T_t(I)$. The procedure may be repeated until the elements of $T_t(I)$ found in this way generate an ideal in $C(\boldsymbol{x})\langle \boldsymbol{\partial_x} \rangle$ whose dimension matches the dimension predicted by Thm. 3.

Example 7. *Applying the algorithm to the ideal I of Example 4 with respect to k, we obtain a coupled system of size 5×14 when A and B are assumed to have total degree 2 and 1, respectively. This system has no solution. For total degrees 3 and 2, respectively, the obtained system is of size 14×28 and has the nontrivial solution*

$$A = S_m + S_l + (2 + l + m)S_l S_m - S_l S_m S_n,$$

$$B = \frac{k(k+1)}{k^2 - 1 - n - kn} S_l + \frac{(m+1)k}{k - n - 1} S_m S_l,$$

hence $A + (S_k - 1)B \in I$, and $A \in T_k(I)$.

6. FINAL COMMENTS

We have seen that algorithms for special functions doing computations in Ore algebras are not restricted to special functions with ∂-finite annihilating ideals. Instead, both closure properties algorithms and algorithms for definite integration and summation can be formulated for arbitrary ideals. The treatment could be extended further by including, for instance, ideals of Laurent Ore algebras.

Our generalized algorithms rely on the notion of ideal dimension as well as on the notion of polynomial growth we introduced in Definition 3. According to this definition, the polynomial growth depends on the monomial order imposed on the underlying algebra. Future research will focus on reducing the notion of polynomial growth to an intrinsic property of the ideal at hand, as well as to devising an algorithm for computing the polynomial growth.

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