A Generalized Apagodu-Zeilberger Algorithm

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ABSTRACT

The Apagodu-Zeilberger algorithm can be used for computing annihilating operators for definite sums over hypergeometric terms, or for definite integrals over hyperexponential functions. In this paper, we propose a generalization of this algorithm which is applicable to arbitrary ∂-finite functions. In analogy to the hypergeometric case, we introduce the notion of proper ∂-finite functions. We show that the algorithm always succeeds for these functions, and we give a tight a priori bound for the order of the output operator.

Categories and Subject Descriptors

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General Terms

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Symbolic summation, symbolic integration, ∂-finite function, holonomic function, Ore algebra, creative telescoping

1. INTRODUCTION

We consider the problem of creative telescoping: given a function $f(x, y)$, the task consists in finding a linear operator $T \neq 0$ in x only, called a *telescoper*, and another operator C possibly involving both x and y, called a *certificate* for T, such that $T - \partial_y C$ annihilates the given function $f(x, y)$. Here ∂_y may be for example the partial derivation $\frac{\mathrm{d}}{\mathrm{d}y}$ or the forward difference operator Δ_y with respect to y.

Pairs (T, C) are used for solving summation and integration problems. For example, given a definite integral $F(x) = \int_0^1 f(x, y)\,\mathrm{d}y$ depending on a free parameter x, we may want to compute a creative telescoping relation

$$\left(t_0 + t_1\partial_x + \cdots + t_r\partial_x^r\right) \cdot f(x, y) = \partial_y C \cdot f(x, y),$$

where ∂_x, ∂_y are the partial derivations $\frac{\mathrm{d}}{\mathrm{d}x}, \frac{\mathrm{d}}{\mathrm{d}y}$, respectively. By integrating both sides of the relation above w.r.t. y, we obtain an inhomogeneous linear differential equation

$$t_0F(x) + t_1F'(x) + \cdots + t_rF^{(r)}(x) = \left[\, C \cdot f(x, y)\,\right]_{y=0}^{1}$$

for the integral. This equation can then be processed further by other algorithms, for example to find closed form representations or asymptotic expansions for $F(x)$.

Algorithms for computing creative telescoping pairs (T, C) are known for various classes of functions $f(x, y)$. For hypergeometric terms, which satisfy two first-order recurrence equations in x and y respectively, the problem is solved by Zeilberger's algorithm [22, 23]. An analogous algorithm for hyperexponential functions, which satisfy two first-order differential equations in x and y respectively, was given by Almkvist and Zeilberger [3]. In 1997, Chyzak [11] generalized these algorithms to the case of general holonomic ∂-finite functions $f(x, y)$, which are solutions of systems of higher-order recurrence and/or differential equations, see Section 2.2 for a definition. For a detailed introduction to creative telescoping in the context of holonomic functions, see [16, 12].

In 2005, Apagodu (formerly "Mohamud Mohammed") and Zeilberger [17] proposed an interesting variation of Zeilberger's original algorithm for hypergeometric terms. This algorithm, sketched in Section 2.4 below, is easier to implement than Zeilberger's original algorithm, it requires less computation time, and it gives rise to good bounds for the order of the telescopers. A similar approach to compute telescopers for general holonomic ∂-finite functions was proposed and implemented in [15]; it proved superior to Chyzak's algorithm in many examples from applications, but used some heuristics and thus lacked rigor. In particular, no bounds concerning the telescoper were given there.

In the present paper, we want to do with the Apagodu-Zeilberger algorithm what Chyzak did with the original Zeilberger algorithm: we extend it to a more general setting. The setting is more general in two senses. First, we drop the condition that the input is specified by first-order equations and instead cover arbitrary ∂-finite input. Second, we do not restrict to the shift and/or differential case but formulate the result in the language of Ore algebras. In this general context, we lose the property known for the differential case that a creative telescoping pair (T, C) always ex-

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ists. Therefore, in analogy to the hypergeometric case, we introduce the notion of *proper ∂-finite* functions, and give an explicit upper bound on the order of telescopers for such functions. Good bounds are useful in practice as they allow to compute telescoper and certificate without having to loop over the order of the ansatz operator (as it is done, for example, in Zeilberger's algorithm).

2. PRELIMINARIES

2.1 Ore Algebras

The operator algebras we are going to work with were introduced by Ore in 1933 [18]. They provide a common framework for representing linear differential equations and linear (q-) difference equations; the coefficients of these equations may be polynomials or rational functions, for example.

Let K be a field with $\mathbb{Q} \subseteq K$. Let $\sigma_x, \sigma_y \colon K(x,y) \to K(x,y)$ be field automorphisms with $\sigma_x\sigma_y = \sigma_y\sigma_x$, and let $\delta_x, \delta_y \colon K(x,y) \to K(x,y)$ be K-linear maps satisfying $\delta_x(ab) = \delta_x(a)b + \sigma_x(a)\delta_x(b)$ and $\delta_y(ab) = \delta_y(a)b + \sigma_y(a)\delta_y(b)$ for all $a, b \in K(x,y)$. The set $\mathbb{A} = K(x,y)[\partial_x, \partial_y]$ of all bivariate polynomials in ∂_x, ∂_y with the usual addition, and with the unique noncommutative multiplication satisfying $\partial_x\partial_y = \partial_y\partial_x$ and $\partial_x a = \sigma_x(a)\partial_x + \delta_x(a)$ and $\partial_y a = \sigma_y(a)\partial_y + \delta_y(a)$ for all $a \in K(x,y)$ is an *Ore algebra* [13]. All Ore algebras appearing in this paper will be of this form.

Note that $\delta_x(\frac{1}{a}) = -\frac{\delta_x(a)}{a\sigma_x(a)}$ for all $a \in K(x,y) \setminus \{0\}$, and likewise for δ_y.

We assume that $\sigma_x, \sigma_y, \delta_x, \delta_y$ map polynomials to polynomials. Moreover, we assume that $\deg_x(\sigma_x(p)) = \deg_x(p)$, $\deg_y(\sigma_x(p)) = \deg_y(p)$, $\deg_x(\delta_x(p)) \le \deg_x(p) - 1$ and that $\deg_y(\delta_x(p)) \le \deg_y(p)$ for all $p \in K[x,y]$; likewise for σ_y, δ_y.

2.2 ∂-Finite Functions

Many special functions used in mathematics and physics are solutions of systems of linear differential and/or recurrence equations. Hypergeometric terms are functions that satisfy a system of first-order linear recurrence equations and their continuous analogue are hyperexponential functions. Their generalization to functions that satisfy a system of higher-order equations leads to the concept of ∂-finite functions.

We let the Ore algebra $\mathbb{A}$ act on an appropriate space F of "functions" by defining an operation $\cdot \colon \mathbb{A} \times F \to F$; in particular, one has to fix the result of applying ∂_x and ∂_y to a function. The operation of applying an Ore operator $P \in \mathbb{A}$ to a function $f \in F$ turns F into a left $\mathbb{A}$-module. We define the *annihilator* (w.r.t. $\mathbb{A}$) of a function f as the set $\{P \in \mathbb{A} \mid P \cdot f = 0\}$, denoted $\operatorname{ann}_{\mathbb{A}}(f)$; it is easy to verify that it is a left ideal in $\mathbb{A}$. For every left ideal $\mathfrak{a} \subseteq \mathbb{A}$ the quotient algebra $\mathbb{A}/\mathfrak{a}$ is a $K(x,y)$-vector space.

A left ideal $\mathfrak{a} \subseteq \mathbb{A}$ is called *zero-dimensional* or *∂-finite* if $\dim_{K(x,y)}(\mathbb{A}/\mathfrak{a})$ is finite. A function f is called *∂-finite* (w.r.t. $\mathbb{A}$) if $\operatorname{ann}_{\mathbb{A}}(f)$ is a zero-dimensional left ideal.

2.3 Left and Right Borders

Part of the additional generality provided in this paper comes at the expense of a somewhat involved notation, which we now introduce.

For $a \in K(x,y)$ and $i \in \mathbb{N}$, write $(a;i)_y := \prod_{j=0}^{i-1} \sigma_y^j(a)$. Let p be a polynomial in $K(x)[y] \setminus \{0\}$. Choose n to be the largest nonnegative integer such that there is a monic factor p_n in $K(x)[y] \setminus K(x)$ with $\deg_y(p_n)$ as large as possible and $(p_n;n)_y$ dividing p. We repeat this process for $p/(p_n;n)_y$ until obtaining a constant $c \in K(x)$. In this way, the polynomial p can be written uniquely as $p = c\prod_{i=1}^n (p_i;i)_y$ with $c \in K(x) \setminus \{0\}$ and $p_1, \ldots, p_n \in K(x)[y]$ monic such that $\deg_y(p_n) > 0$. When $\sigma_y = \mathrm{id}$ this is the squarefree decomposition of p in y, and when $\sigma_y(y) = y+1$ it is the greatest factorial factorization [19] in y, where the falling factorials in the original definition are expressed in terms of rising factorials. Define $p\lceil_y := \prod_{i=1}^n p_i$ (left border of p) and $p\rceil_y := \prod_{i=1}^n \sigma_y^{i-1}(p_i)$ (right border of p). When $\sigma_y = \mathrm{id}$, both the left border $p\lceil_y$ and the right border $p\rceil_y$ are equal to the squarefree part of p. By definition, we have $p\,\sigma_y(p\rceil_y) = c\prod_{i=1}^n (p_i; i+1)_y$, $\frac{p}{p\rceil_y} = c\prod_{i=1}^n (p_i; i-1)_y$, and the equality

$$p\,\sigma_y(p\rceil_y) = p\lceil_y\,\sigma_y(p). \tag{1}$$

The notations $(a;i)_x$ (for $a \in K(x,y)$) and $p\lceil_x, p\rceil_x$ (for $p \in K(y)[x] \setminus \{0\}$) are defined analogously.

Lemma 1. *For $a \in K(x,y) \setminus \{0\}$ we have*

$$\delta_y\big((a;n)_y\big) = \frac{(a;n)_y}{a} \sum_{i=0}^{n-1} \delta_y\big(\sigma_y^i(a)\big). \tag{2}$$

Moreover, for $p \in K(x)[y] \setminus \{0\}$ we have that $p \mid p\lceil_y \delta_y(p)$, as polynomials in y. Analogous statements hold when switching the roles of x and y.

Proof. The proof uses the general product rule for δ_y,

$$\delta_y\Big(\prod_{i=1}^n a_i\Big) = \sum_{i=1}^n \Big(\prod_{j=1}^{i-1} \sigma_y(a_j)\Big)\delta_y(a_i)\Big(\prod_{j=i+1}^n a_j\Big), \tag{3}$$

where $a_1, \ldots, a_n$ are arbitrary elements in $K(x,y)$; the formula can be verified by an easy induction argument. Applying (3) to $(a;n)_y$ proves the first assertion:

$$\begin{aligned}
\delta_y\big((a;n)_y\big) &= \delta_y\Big(\prod_{i=1}^n \sigma_y^{i-1}(a)\Big) \\
&= \sum_{i=1}^n \Big(\prod_{j=1}^{i-1} \sigma_y^j(a)\Big)\delta_y\big(\sigma_y^{i-1}(a)\big)\Big(\prod_{j=i+1}^n \sigma_y^{j-1}(a)\Big) \\
&= (\sigma_y(a); n-1)_y \sum_{i=0}^{n-1} \delta_y\big(\sigma_y^i(a)\big)
\end{aligned}$$

To prove the second assertion, assume that p is given in the unique form $c\prod_{i=1}^n (p_i;i)_y$ as above. Once again, the product rule (3) is employed, in combination with the first assertion:

$$\begin{aligned}
\delta_y(p) &= \delta_y\Big(c\prod_{i=1}^n (p_i;i)_y\Big) \\
&= c\sum_{i=1}^n \Big(\prod_{j=1}^{i-1} \sigma_y\big((p_j;j)_y\big)\Big)\delta_y\big((p_i;i)_y\big)\Big(\prod_{j=i+1}^n (p_j;j)_y\Big) \\
&= \frac{c\,p}{p\lceil_y} \sum_{i=1}^n \Big(\prod_{j=1}^{i-1} \sigma_y^j(p_j)\Big)\Big(\sum_{k=0}^{i-1} \delta_y\big(\sigma_y^k(p_i)\big)\Big)\Big(\prod_{j=i+1}^n p_j\Big).
\end{aligned}$$

Note that $\frac{p}{p\lceil_y} = \prod_{j=2}^n \sigma_y\big((p_j;j-1)_y\big)$ in the last line. This completes the proof. ∎

2.4 The Apagodu-Zeilberger Algorithm

The Apagodu-Zeilberger algorithm [17] solves the same problem as Zeilberger's algorithm [23]: creative telescoping for proper hypergeometric terms. Before generalizing this algorithm to general ∂-finite functions, let us summarize the reasoning behind it at a simple example. Consider the hypergeometric term $h(x,y) := \frac{1}{\Gamma(ax+by)}$ for two positive integers $a,b \in \mathbb{N}$. We want to find $T = t_0 + t_1\partial_x + \cdots + t_r\partial_x^r \in K(x)[\partial_x] \setminus \{0\}$ (a *telescoper*) and $C \in K(x,y)$ (a *certificate*) such that

$$T \cdot h(x,y) = \partial_y C \cdot h(x,y),$$

where ∂_x denotes the shift operator with respect to x (i.e. $\sigma_x(x) = x+1$, $\delta_x = 0$) and ∂_y denotes the forward difference with respect to y (i.e. $\sigma_y(y) = y+1$, $\delta_y(y) = 1$).

By $\Gamma(ax+by+ia) = (ax+by)(ax+by+1)\cdots(ax+by+ia-1)\cdot\Gamma(ax+by)$ for all $i \geq 0$, we have

$$T \cdot h(x,y) = \frac{u}{(ax+by)\cdots(ax+by+ra-1)}h(x,y)$$

for some polynomial u of y-degree ra whose coefficients are linear combinations of the undetermined coefficients t_i. For the choice

$$C = \frac{c_0 + c_1 y + \cdots + c_s y^s}{(ax+by)(ax+by+1)\cdots(ax+by+ra-b-1)},$$

we obtain

$$\partial_y C \cdot h(x,y) = \frac{v}{(ax+by)\cdots(ax+by+ra-1)}h(x,y)$$

for some polynomial v of y-degree $s+b$. The denominators on both sides agree, and if we take $s = ra-b$, so do the degrees (provided that $ra \geq s$). Coefficient comparison yields a linear homogeneous system with $ra+1$ equations and $(r+1)+(ra-b+1)$ variables (the t_i's and the c_j's). As soon as $r \geq b$, this system has a nontrivial solution.

A telescoper T coming from such a nontrivial solution cannot be zero, for if it were, then also $\partial_y C \cdot h(x,y)$ would be zero, and then $C \cdot h(x,y)$ would be constant with respect to y, which is not the case because C is a nonzero rational function but $h(x,y)$ is not rational.

Similar calculations can be carried out for when a or b are negative. By plugging all of them together, Apagodu and Zeilberger [17] show that a (non-rational) proper hypergeometric term

$$p\,\alpha^x\beta^y \prod_{m=1}^{M} \frac{\Gamma(a_m x + a'_m y + a''_m)\Gamma(b_m x - b'_m y + b''_m)}{\Gamma(u_m x + u'_m y + u''_m)\Gamma(v_m x - v'_m y + v''_m)}$$

($p \in K[x,y]$, $M \in \mathbb{N}$, $\alpha, \beta, a''_m, b''_m, u''_m, v''_m \in K$, a_m, a'_m, b_m, b'_m, u_m, u'_m, v_m, $v'_m \in \mathbb{N}$) admits a nonzero telescoper of order at most $\max\{\sum_{m=1}^{M}(a'_m + v'_m), \sum_{m=1}^{M}(u'_m + b'_m)\}$. A refinement of this bound, including the x-degree of the telescoper, is given in [7].

In the differential case, they find [4] that a (non-rational) hyperexponential function

$$p\,\exp\Big(\frac{a}{b}\Big)\prod_{m=1}^{M} q_m^{e_m}$$

($a, b, p, q_1, \ldots, q_M \in K(x)[y]$, $e_1, \ldots, e_m \in K$) admits a telescoper of order at most $\deg_y(b) + \max\{\deg_y(a), \deg_y(b)\} + \sum_{m=1}^{M}\deg_y(q_m) - 1$. In [8, Thm. 14] it is shown that this bound can be improved by replacing the first term $\deg_y(b)$ by the y-degree of the squarefree part of b, and that when the term is a rational function the bound increases by 1. A further improvement is given in [5, Sec. 6.2].

3. THE GENERAL CASE

Let $\mathbb{A} = K(x,y)[\partial_x, \partial_y]$ be an Ore algebra as introduced in Section 2.1 and $\mathfrak{a} \subseteq \mathbb{A}$ be a ∂-finite ideal. Further let $B = \{b_1, \ldots, b_n\}$ be a $K(x,y)$-basis of $\mathbb{A}/\mathfrak{a}$, so that every element of $\mathbb{A}/\mathfrak{a}$ can be written uniquely in the form $wb = \sum_{i=1}^{n} w_i b_i$ for some vector $w = (w_1, \ldots, w_n) \in K(x,y)^n$ and $b = (b_1, \ldots, b_n)^T$. We say that the vector $w = (w_1, \ldots, w_n)$ represents the element $wb \in \mathbb{A}/\mathfrak{a}$. For all $b_i \in B$ we can write

$$\partial_x b_i = \sum_{j=1}^{n} m_{i,j} b_j \quad \text{with } m_{i,j} \in K(x,y). \tag{4}$$

With $M = (m_{i,j})_{1\leq i,j\leq n} \in K(x,y)^{n\times n}$ equation (4) can be expressed succinctly as $\partial_x b = Mb$, where the operator ∂_x is applied componentwise. Applying ∂_x to an arbitrary element in $\mathbb{A}/\mathfrak{a}$ gives

$$\partial_x(wb) = \big(\sigma_x(w)\partial_x + \delta_x(w)\big)b = \big(\sigma_x(w)M + \delta_x(w)\big)b$$

where σ_x and δ_x act on the components of $w \in K(x,y)^n$. As a shorthand notation, we will write the above identity as $\partial_x w = \sigma_x(w)M + \delta_x(w)$, by defining an action of operators from $\mathbb{A}$ on vectors in $K(x,y)^n$. Similarly, there is a matrix $N \in K(x,y)^{n\times n}$ such that $\partial_y b = Nb$ and $\partial_y w = \sigma_y(w)N + \delta_y(w)$. Without loss of generality, we assume that the basis B is chosen in such a way that the element $1 \in \mathbb{A}/\mathfrak{a}$ is represented by a *polynomial* vector $e \in K(x)[y]^n$; we call such bases (y-)*admissible*.

The matrices M and N correspond to the rational functions $\partial_x h/h$ and $\partial_y h/h$ in the hyperexponential case, and similarly in the hypergeometric case. In general, their entries are rational functions. We will write $M = \frac{1}{u}U$, $N = \frac{1}{v}V$ where $u, v \in K[x,y]$ and $U, V \in K[x,y]^{n\times n}$.

3.1 Telescoper Part

For $r \geq 1$, make an ansatz $T = \sum_{i=0}^{r} t_i \partial_x^i$ for the telescoper, in which $t_0, \ldots, t_r$ stand for undetermined elements of $K(x)$, so that T is an element of $K(x)[\partial_x] \subseteq \mathbb{A}$. We need to discuss the shape of the vector $Te \in K(x,y)^n$, i.e., its denominator and its numerator degree in y.

Lemma 2. *Let $e \in K(x)[y]^n$ be some polynomial vector. For every $i \geq 0$ we have $\partial_x^i e = \frac{1}{(u;i)_x}w$ for some vector $w \in K(x)[y]^n$ with*

$$\deg_y(w) \leq \deg_y(e) + i\max\{\deg_y(u), \deg_y(U)\},$$

where the degree of a matrix or vector refers to the maximum degree of its components.

Proof. The claim is evident for $i = 0$. Assume it holds for i. Then

$$\begin{aligned}
\partial_x^{i+1} e &= \partial_x\Big(\frac{1}{(u;i)_x}w\Big) \\
&= \sigma_x\Big(\frac{1}{(u;i)_x}w\Big)\frac{1}{u}U + \delta_x\Big(\frac{1}{(u;i)_x}w\Big) \\
&= \frac{\sigma_x(w)U}{u\,\sigma_x\big((u;i)_x\big)} + \frac{\delta_x(w)}{(u;i)_x} + \delta_x\Big(\frac{1}{(u;i)_x}\Big)\sigma_x(w) \\
&= \frac{\sigma_x(w)U}{(u;i+1)_x} + \frac{\delta_x(w)}{(u;i)_x} - \frac{\delta_x\big((u;i)_x\big)\,\sigma_x(w)}{(u;i)_x\,\sigma_x\big((u;i)_x\big)}.
\end{aligned}$$

The last term in the above line can be simplified by

$$\frac{\delta_x\big((u;i)_x\big)}{(u;i)_x\,\sigma_x\big((u;i)_x\big)} = \frac{1}{(u;i+1)_x}\frac{u\,\delta_x\big((u;i)_x\big)}{(u;i)_x} = \frac{\tilde u}{(u;i+1)_x}$$

where Lemma 1 ensures that $\tilde u$ is a polynomial in $K[x,y]$. Since we assume throughout that σ_x and δ_x do not increase the y-degree of polynomials, we conclude from (2) that $\deg_y(\tilde u) \le \deg_y(u)$. Therefore, we obtain

$$\partial_x^{i+1} e = \frac{\sigma_x(w)U + \sigma_x^i(u)\delta_x(w) - \tilde u \sigma_x(w)}{(u;i+1)_x},$$

and the whole numerator is bounded in y-degree by

$$\begin{aligned}
&\max\{\deg_y(\sigma_x(w)) + \deg_y(U),\\
&\qquad \deg_y(\sigma_x^i(u)) + \deg_y(\delta_x(w)),\\
&\qquad \deg_y(\tilde u) + \deg_y(\sigma_x(w))\}\\
&\le \max\{\deg_y(w) + \deg_y(U), \deg_y(u) + \deg_y(w),\\
&\qquad \deg_y(u) + \deg_y(w)\}\\
&\le \max\{\deg_y(U), \deg_y(u)\} + \deg_y(w)\\
&\le \deg_y(e) + (i+1)\max\{\deg_y(u), \deg_y(U)\},
\end{aligned}$$

as claimed. ∎

By writing $\frac{1}{(u;i)_x} = \frac{(\sigma^i(u);r-i)_x}{(u;r)_x}$ in the above lemma, we find that we can write

$$Te = \frac{1}{(u;r)_x} w$$

for some vector w whose entries are linear combinations of $t_0,\dots,t_r$ with coefficients in $K(x)[y]$ bounded in degree by $\deg_y(e) + r\max\{\deg_y(u), \deg_y(U)\}$.

3.2 Certificate Part

We need to characterize those certificates $C \in \mathbb{A}$ for which the vector $\partial_y Ce$ matches a prescribed numerator degree and a prescribed denominator $d \in K(x)[y]$; as before, let e denote the polynomial vector representing the element $1 \in \mathbb{A}/\mathfrak{a}$ with respect to the basis B. It will be convenient to focus on possible numerators and denominators of the vector $c := Ce \in K(x,y)^n$.

Let $d \in K(x)[y]$ be the target denominator. It will turn out that factors of d which also appear in v (the denominator in the ∂_y-multiplication matrix) behave slightly different than other factors. Let us therefore write

$$\begin{aligned}
d &= (f_1;p_1)_y \cdots (f_m;p_m)_y\, g\\
v &= (f_1;q_1)_y \cdots (f_m;q_m)_y\, \sigma_y(h)
\end{aligned}$$

so that $f_1,\dots,f_m \in K(x)[y]$ are common factors of d and v. Note that we don't impose any coprimeness conditions on the f_i's with g and h. Therefore, without loss of generality, we may always assume that $p_i \ge q_i$, by moving possible overhanging factors of some $(f_i;q_i)_y$ into $\sigma_y(h)$.

Lemma 3. *Assume that $p_i \ge q_i \ge 1$ for $i = 1,\dots,m$ and let*

$$z = \sigma_y^{-1}\Big(\frac{(f_1;p_1)_y\cdots(f_m;p_m)_y}{(f_1;q_1)_y\cdots(f_m;q_m)_y}\Big)\frac{g}{g\rceil_y}.$$

Note that $z \in K(x)[y]$. Let $w \in K(x)[y]^n$ be any polynomial vector and consider $c = \frac{h}{z}w$. Then $\partial_y c = \frac{1}{d}\tilde w$ for some vector $\tilde w \in K(x)[y]^n$ with $\deg_y(\tilde w) \le \deg_y(w) + \deg_y(g\lceil_y) + \max\{\deg_y(v) - 1, \deg_y(V)\}$.

Proof. We show that $d\,\partial_y c$ is a polynomial vector with the claimed degree. From

$$\partial_y c = \partial_y \frac{h}{z} w = \frac{\sigma_y(h)}{\sigma_y(z)}\sigma_y(w)\frac{1}{v}V + \delta_y\Big(\frac{h}{z}w\Big)$$

we get

$$d\,\partial_y c = \frac{d\,\sigma_y(h)}{v\,\sigma_y(z)}\sigma_y(w)V + d\,\delta_y\Big(\frac{h}{z}\Big)w + \frac{d\,\sigma_y(h)}{\sigma_y(z)}\delta_y(w). \quad (5)$$

A straightforward calculation using (1) gives the equality $d\sigma_y(h) = \sigma_y(z)v\,g\lceil_y$ which we employ in the following to replace d. The first term in expression (5) simplifies to $g\lceil_y\sigma_y(w)V$, the y-degree of which is bounded by $\deg_y(g\lceil_y) + \deg_y(w) + \deg_y(V)$, as claimed. Similarly, the third term simplifies to $v\,g\lceil_y\delta_y(w)$, the y-degree of which is bounded by $\deg_y(v) + \deg_y(g\lceil_y) + \deg_y(w) - 1$, also as claimed. Finally, the second term of (5) is considered:

$$\begin{aligned}
d\,\delta_y\Big(\frac{h}{z}\Big)w &= \frac{\sigma_y(z)\,v\,g\lceil_y}{\sigma_y(h)}\,\frac{\delta_y(h)\,z - h\,\delta_y(z)}{z\,\sigma_y(z)}\,w\\
&= \frac{v}{\sigma_y(h)}\,g\lceil_y\delta_y(h)w - \frac{v\,\delta_y(z)\,g\lceil_y}{z\,\sigma_y(h)}hw.
\end{aligned}$$

The first term in this expression is a polynomial because $\sigma_y(h) \mid v$. Its degree is bounded by $\big(\deg_y(v) - \deg_y(h)\big) + \big(\deg_y(h) - 1\big) + \deg_y(g\lceil_y) + \deg_y(w) = \deg_y(w) + \deg_y(g\lceil_y) + \deg_y(v) - 1$, as claimed. Also for the second term the degree count matches the claim. To see finally that also this second term is a polynomial in y, write

$$z = \Big(\prod_{i=1}^m (\sigma_y^{q_i-1}(f_i); p_i - q_i)_y\Big) g,$$

which implies that $z\lceil_y$ divides $(f_1;q_1)_y\cdots(f_m;q_m)_y\,g\lceil_y$. We now write the second term as

$$\frac{v\,\delta_y(z)\,g\lceil_y}{z\,\sigma_y(h)}hw = \frac{(f_1;q_1)_y\cdots(f_m;q_m)_y\,g\lceil_y}{z\lceil_y}\,\frac{z\lceil_y\,\delta_y(z)}{z}\,hw$$

and observe that z divides $z\lceil_y\,\delta_y(z)$ by Lemma 1, which concludes the proof. ∎

3.3 Proper ∂-finite Ideals

In order to obtain a bound for the order of the telescoper, we apply Lemmas 2 and 3 in such a way that the vectors Te and $\partial_y c$ match. In particular, we need to match the denominator and the degree of the numerator. From Section 3.1 we know that the denominator coming from the telescoper part is $(u;r)_x$, and the y-degree of the numerator is at most $\deg_y(e) + r\max\{\deg_y(u), \deg_y(U)\}$. From Section 3.2 we know how to choose c in such a way that $\partial_y c$ has a prescribed denominator and a given numerator degree. Coefficient comparison with respect to y will give a system of linear equations, and we will be able to choose r in such a way that this system has a solution.

This is the basic idea, but there is a complication. The denominator coming from the telescoper part is expressed with respect to σ_x while Lemma 3 requires the prescribed denominator to be expressed with respect to σ_y. There is of course no difference (and hence no complication) when $\sigma_x = \sigma_y = \mathrm{id}$, as for instance in the differential case. However, in general it is necessary to impose some further assumption on the ∂-finite ideal $\mathfrak{a}$ in order for the argument to go through.

We propose one such assumption in the following definition. It generalizes the distinction between hypergeometric terms and *proper* hypergeometric terms known from classical summation theory [21, 2]. At the same time, it refines this notion by distinguishing the free variable x from the summation/integration variable y.

Definition 4. *1. A polynomial $u \in K[x,y]$ is called* y-proper *with respect to two endomorphisms σ_x, σ_y if $\deg_y\big((u;r)_x\lceil_y\big) = \mathrm{O}(1)$ as $r \to \infty$.*

2. A ∂-finite ideal $\mathfrak{a} \subseteq K(x,y)[\partial_x,\partial_y] =: \mathbb{A}$ is called proper *(with respect to y) if there exists a y-admissible basis B of $\mathbb{A}/\mathfrak{a}$, i.e., the element $1 \in \mathbb{A}/\mathfrak{a}$ is represented by a vector in $K(x)[y]^n$, for which the multiplication matrix $\frac{1}{u}U$ is such that u is y-proper with respect to the two endomorphisms σ_x and σ_y of $\mathbb{A}$.*

3. Let B and $\frac{1}{u}U, \frac{1}{v}V$ be as above. Let $\eta \in \mathbb{N}$ be the smallest number such that for all $r \geq 1$ there exist $f_1,\dots,f_m,g,h \in K[x,y]$, $p_1,\dots,p_m,q_1,\dots,q_m \in \mathbb{N}$, $p_i \geq q_i \geq 1$ for $i=1,\dots,m$, with

$$v = \sigma_y(h)\prod_{i=1}^{m}(f_i;q_i)_y \quad \text{and} \quad (u;r)_x = g\prod_{i=1}^{m}(f_i;p_i)_y$$

and $\deg_y\big(g\lceil_y\big) \leq \eta$. Then

$$\eta + \max\{\deg_y(v)-1, \deg_y(V)\}$$

is called the height *of $\mathfrak{a}$ with respect to the basis B.*

4. Let $\mathfrak{a} \subseteq \mathbb{A}$ be a proper ∂-finite ideal. The height *of $\mathfrak{a}$ is defined as the minimum height of $\mathfrak{a}$ with respect to all admissible bases of $\mathbb{A}/\mathfrak{a}$.*

It is obvious that when $\sigma_x = \sigma_y = \mathrm{id}$, as for instance in the differential case, then every ∂-finite ideal is proper ∂-finite, because in this case $(u;r)_x\lceil_y$ is simply the squarefree part of u, which does not depend on r. We will further show in Proposition 7 below that in the differential case we always have $\eta = 0$. For the shift case, we will show (Proposition 9) that when $\mathfrak{a}$ is the annihilator of a hypergeometric term h, then h is proper hypergeometric if and only if $\mathfrak{a}$ is proper ∂-finite with respect to both x and y.

In part 3 of the definition, observe that the y-properness of u implies that such a number η always exist, because a possible (but perhaps not optimal) choice is $g = (u;r)_x$, $h = \sigma^{-1}(v)$ and no f_i's at all (i.e., $m=0$). The more complicated condition in the definition allows for smaller values of η by discarding common factors of u and v. This is desirable because smaller values of η will lead to a smaller bound for the telescoper in Theorem 6 below.

Example 5. *We demonstrate that the right choice of the basis $B = \{b_1,\dots,b_n\}$ is crucial in the definition of proper ∂-finite ideals. Let $H \in K(x,y)^{n\times n}$ denote the matrix that realizes the change to a new basis $\tilde{B} = \{\tilde{b}_1,\dots,\tilde{b}_n\}$, i.e., $\tilde{b} = Hb$ with $b = (b_1,\dots,b_n)^T$ and $\tilde{b} = (\tilde{b}_1,\dots,\tilde{b}_n)^T$. If M is the ∂_x-multiplication matrix with respect to B, i.e., $\partial_x b = Mb$ as in (4), then from*

$$\partial_x\tilde{b} = \partial_x Hb = \big(\sigma_x(H)M + \delta_x(H)\big)b = \tilde{M}\tilde{b}$$

it follows that $\tilde{M} = \big(\sigma_x(H)M + \delta_x(H)\big)H^{-1}$ is the ∂_x-multiplication matrix with respect to $\tilde{B}$ (and analogously for N and $\tilde{N}$). In the following, let $\sigma_x(x) = x+1$, $\sigma_x(y) = y$, $\sigma_y(x) = x$, $\sigma_y(y) = y+1$, and $\delta_x = \delta_y = 0$.

Consider the classic example of a function that is hypergeometric but not proper: $f = 1/(x^2+y^2)$; its annihilating ideal $\mathfrak{a}$ is generated by $\big((x+1)^2+y^2\big)\partial_x - x^2 - y^2$ and $\big(x^2+(y+1)^2\big)\partial_y - x^2 - y^2$, thus $n=1$. Choosing $1 \in \mathbb{A}/\mathfrak{a}$ as the single basis element b_1, one gets $M = \big(x^2+y^2\big)/\big((x+1)^2+y^2\big)$; its denominator is clearly not y-proper w.r.t. σ_x and σ_y. Performing the basis change $H = x^2+y^2$ yields $\tilde{M} = 1$ whose denominator is y-proper. But still, the new basis $\tilde{B}$ does not certify that f is proper ∂-finite since in $\tilde{B}$ the element $1 \in \mathbb{A}/\mathfrak{a}$ is represented by $e = H^{-1}$, and therefore $\tilde{B}$ is not admissible.

Next consider the function $f = 1/(x+y)! + 1/(x-y)!$ with the standard monomial basis $B = \{1, \partial_y\}$, i.e., the basis elements b_1 and b_2 correspond to $f(x,y)$ and $f(x,y+1)$. With respect to this basis, the matrix M is

$$M = \frac{1}{p}\begin{pmatrix} \frac{x^2-2xy+y^2+x-y-1}{y-x+1} & \frac{2y}{y-x-1} \\ \frac{2(y+1)}{x+y+2} & -\frac{x^2+2xy+y^2+3x+3y+1}{x+y+2} \end{pmatrix}$$

where $p = y^2 - x^2 + y - x + 1$ is an irreducible quadratic factor. Again, the basis B does not certify properness (see also Proposition 9), but this time we succeed in finding an admissible basis $\tilde{B}$ which does. With

$$H = \frac{1}{p}\begin{pmatrix} (y-x)(x+y+1) & x+y+1 \\ 1 & -(x+y+1) \end{pmatrix}$$

the multiplication matrices $\tilde{M} = \sigma_x(H)MH^{-1}$ and $\tilde{N}$ are

$$\tilde{M} = \begin{pmatrix} \frac{1}{x+y+1} & 0 \\ 0 & \frac{1}{x-y+1} \end{pmatrix} = \frac{1}{u}U, \quad \tilde{N} = \begin{pmatrix} \frac{1}{x+y+1} & 0 \\ 0 & x-y \end{pmatrix}.$$

(Note that the diagonal structure of these matrices reveals that the basis elements $\tilde{b}_1$ and $\tilde{b}_2$ correspond to $1/(x+y)!$ and $1/(x-y)!$, respectively.) We have now

$$(u;r)_x = \prod_{i=1}^{r}\big((x+y+i)(x-y+i)\big) = ((x+y+1)(x-y+r);r)_y$$

and therefore $(u;r)_x\lceil_y = (x+y+1)(x-y+r)$ for all $r \geq 1$.

3.4 Main Result

We are now ready to show the existence of telescopers for proper ∂-finite ideals, and to give an explicit bound on their order.

Theorem 6. *Assume that $\mathfrak{a} \subseteq \mathbb{A} = K(x,y)[\partial_x,\partial_y]$ is proper ∂-finite with respect to y. Let ϱ be the height of $\mathfrak{a}$, let $n = \dim_{K(x,y)}\mathbb{A}/\mathfrak{a}$, and let $\phi = \dim_{K(x)}\{W \in \mathbb{A}/\mathfrak{a} \mid \partial_y W = 0\}$. Then there exist $T \in K(x)[\partial_x]\setminus\{0\}$ and $C \in \mathbb{A}$ such that $T - \partial_y C \in \mathfrak{a}$ and $\mathrm{ord}(T) \leq n\varrho + \phi$.*

Proof. Let $r = n\varrho + \phi$ and make an ansatz $T = t_0 + t_1\partial_x + \dots + t_r\partial_x^r$ with undetermined $t_i \in K(x)$ for a telescoper. Let $B, \frac{1}{u}U, \frac{1}{v}V, f_1,\dots,f_m,g,h,q_1,\dots,q_m,p_1,\dots,p_m$ be as in Definition 4. Let $e \in K(x)[y]^n$ be the vector representing $1 \in \mathbb{A}/\mathfrak{a}$ with respect to B, let $\gamma = \max\{\deg_y(u), \deg_y(U)\}$, and let $s = \deg_y(e) + r\gamma - \varrho$. For the certificate part make an ansatz $c = \frac{h}{z}\big(\sum_{j=0}^{s} c_{1,j}y^j, \dots, \sum_{j=0}^{s} c_{n,j}y^j\big)$ with z as in Lemma 3 and undetermined coefficients $c_{i,j} \in K(x)$. A corresponding operator $C \in \mathbb{A}$ with $Ce = c$ is obtained by $C = c\cdot(B_1,\dots,B_n)^T$ where B_i is an operator in $\mathbb{A}$ such that $B_i \cdot 1_{\mathbb{A}/\mathfrak{a}} = b_i$.

According to Lemma 2, $Te = \frac{1}{(u;r)_x} w_T$ for some vector w_T whose entries are linear combinations of the undetermined t_i with coefficients in $K(x)[y]$ of degree at most $\deg_y(e)+r\gamma$.

According to Lemma 3, $\partial_y c = \frac{1}{(u;r)_x} w_C$ for some vector w_C whose entries are linear combinations of the undetermined $c_{i,j}$ with coefficients in $K(x)[y]$ of degree at most $s+\varrho = \deg_y(e) + r\gamma$.

Comparing coefficients with respect to y in all the n coordinates of w_T and w_C gives a linear system over $K(x)$ with $n(\deg_y(e)+r\gamma+1)$ equations in $(r+1)+n(s+1)$ unknowns. This system has a solution space of dimension at least

$$\begin{aligned}&(r+1)+n(s+1)-n(\deg_y(e)+r\gamma+1)\\&=(r+1)+n(\deg_y(e)+r\gamma-\varrho+1)-n(\deg_y(e)+r\gamma+1)\\&=r+1-n\varrho=\phi+1.\end{aligned}$$

As this is greater than ϕ, the solution space must contain at least one vector which corresponds to a nonzero operator T. ∎

Note that the number ϕ in Theorem 6 is bounded by n. To see this, write $W = \sum_{i=1}^n w_i b_i = wb$ for some undetermined $w_i \in K(x,y)$. Then the requirement $\partial_y W = 0$ translates into a first-order linear system of functional equations $\sigma_y(w)\frac{1}{v}V + \delta_y(w) = 0$. It is well known that such a system can have at most n solution vectors that are linearly independent over the field of σ_y-constants, which is $K(x)$ in our case. For a hypergeometric term $h(x,y)$ we have that $\phi = 1$ if and only if $h \in K(x,y)$; this explains why Theorem 6 doesn't exclude such special cases, as opposed to Apagodu and Zeilberger's theorem, see Section 2.4.

Theorem 6 also contains an algorithm for creative telescoping, at least when ϱ (and the corresponding basis B) and ϕ are known or can be computed. In this case, it suffices to make an ansatz for telescoper and certificate as in the proof, compare coefficients, and then solve the resulting linear system.

4. IMPORTANT SPECIAL CASES

Most important in applications are the differential case (integration) and the shift case (summation). We will discuss the implications of Definition 4 and Theorem 6 for these two cases. Whether a ∂-finite ideal is proper or not depends mostly on the denominators u and v of the multiplication matrices, and not so much on the numerators U and V. Since u and v are not matrices but only scalar polynomials, the following discussion is not much different from the hyperexponential or hypergeometric case.

4.1 Differential Case

We consider the case where we act on both variables x and y with the partial derivation, i.e., we have $\sigma_x = \sigma_y = \mathrm{id}$, $\delta_x = \frac{\mathrm{d}}{\mathrm{d}x}$, and $\delta_y = \frac{\mathrm{d}}{\mathrm{d}y}$. We have already mentioned that in the differential case every ∂-finite ideal is proper ∂-finite. We now show that in this case u and v must be essentially equal. This generalizes Lemma 8 of [14]. A consequence is that in part 3 of Definition 4 we can always take $\eta = 0$.

Proposition 7. *If $\mathfrak{a} \subseteq \mathbb{A}$ is ∂-finite, B is a basis of $\mathbb{A}/\mathfrak{a}$ and the multiplication matrices are $\frac{1}{u}U, \frac{1}{v}V$, then the squarefree part of u in $K(x)[y]$ divides the squarefree part of v in $K(x)[y]$.*

Proof. Let $M := \frac{1}{u}U$ and $N := \frac{1}{v}V$. By definition,

$$\begin{aligned}\partial_x\partial_y w &= \partial_x(wN + \delta_y(w))\\&= wNM + \delta_x(wN) + \delta_y(w)M + \delta_x\delta_y(w)\\&= wNM + w\delta_x(N) + \delta_x(w)N + \delta_y(w)M + \delta_x\delta_y(w).\\ \partial_y\partial_x w &= \partial_y(wM + \delta_x(w))\\&= wMN + \delta_y(wM) + \delta_x(w)N + \delta_y\delta_x(w)\\&= wMN + w\delta_y(M) + \delta_y(w)M + \delta_x(w)N + \delta_y\delta_x(w).\end{aligned}$$

Because of $\partial_x\partial_y = \partial_y\partial_x$, we have the compatibility condition

$$NM + \delta_x(N) = MN + \delta_y(M).$$

Let $p \in K[x,y]$ with $\deg_y(p) > 0$ be an irreducible factor of u, let (i,j) be such that $p \nmid U_{i,j}$, and let m be the multiplicity of p in u. Then the multiplicity of p in the denominator of $\delta_y(\frac{1}{u}U)_{i,j}$ is $m+1$. If p were not also a factor of v, then the multiplicity of p in the denominator of $\frac{1}{v}V\frac{1}{u}U + \delta_x(\frac{1}{v}V) - \frac{1}{u}U\frac{1}{v}V$ could be at most m. ∎

Example 8. *Let*

$$p = (x^2+x+1) + (2x^2-x+1)y + (x^2-2x+3)y^2$$

and let $\mathfrak{a} \subseteq \mathbb{Q}(x,y)[\partial_x,\partial_y]$ be the annihilator of $f = p^{-1/3} + p^{-1/5}$. Then $n = \dim_{\mathbb{Q}(x,y)} \mathbb{A}/\mathfrak{a} = 2$ (since the two summands of f are hyperexponential but $\mathbb{Q}(x,y)$-linearly independent) and $\phi = \dim_{\mathbb{Q}(x)}\{W \in \mathbb{A}/\mathfrak{a} \mid \partial_y W = 0\} = 0$. The algebra $\mathbb{A}/\mathfrak{a}$ is isomorphic as $K(x,y)$-vector space to $K(x,y)p^{-1/3} + K(x,y)p^{-1/5}$. With respect to the basis $B = \{p^{-1/3}, p^{-1/5}\}$, the element $1 \in \mathbb{A}/\mathfrak{a}$ is represented by the vector $(1,1) \in \mathbb{K}(x)[y]^2$ and the multiplication matrices are

$$M = \frac{D_x(p)}{p}\begin{pmatrix}-1/3 & 0\\ 0 & -1/5\end{pmatrix},\quad N = \frac{D_y(p)}{p}\begin{pmatrix}-1/3 & 0\\ 0 & -1/5\end{pmatrix}.$$

We can therefore take $u = v = p$ and have

$$\max\{\deg_y(v)-1, \deg_y(V)\} = 1.$$

Theorem 6 predicts a telescoper of order $1\cdot 2+0 = 2$, and it can be confirmed for instance using Chyzak's algorithm that this is in fact the minimal order operator.

Repeating a similar calculation with random polynomials p of y-degree d ($d = 2,\dots,5$) and linear combinations $f = p^{e_1} + \cdots + p^{e_n}$ with n rational exponents with pairwise coprime denominators ($n = 1,\dots,4$), we found the minimal telescopers to be of order $n(d-1)$, in accordance with the bound given in Theorem 6.

In the hyperexponential case, Theorem 6 reduces to the known bound quoted at the end of Section 2.4.

4.2 Shift Case

In this section, let σ_x and σ_y denote the standard shifts with respect to x and y, respectively, i.e., $\sigma_x(x) = x+1$, $\sigma_x(y) = y$, $\sigma_y(x) = x$, $\sigma_y(y) = y+1$. Let δ_y be the forward difference with respect to y and δ_x either identically zero or the forward difference with respect to x.

For a polynomial $p \in K[x,y]$ and $n \in \mathbb{N}$, we write $p^{\overline{n}} := p(p+1)\cdots(p+n-1)$ and $p^{\underline{n}} := p(p-1)\cdots(p-n+1)$. Note that these quantities are in general different from $(p;n)_x$ and $(p;n)_y$.

Proposition 9. *A ∂-finite ideal $\mathfrak{a}$ is proper if and only if there exists an admissible basis B of $\mathbb{A}/\mathfrak{a}$ for which the multiplication matrices $\frac{1}{u}U$, $\frac{1}{v}V$ are such that u is a product of integer-linear polynomials.*

More specifically, suppose we can write

$$u = \prod_{i=1}^{k+\ell} (a_i x + b_i y + c_i)^{\overline{a_i}} (a'_i x - b'_i y + c'_i)^{\underline{a'_i}}$$

$$v = \sigma_y(h) \prod_{i=1}^{k} (a_i x + b_i y + c_i)^{\overline{b_i}} (a'_i x - b'_i y + c'_i)^{\underline{b'_i}}$$

for certain $a_i, b_i, a'_i, b'_i \in \mathbb{N}$ and $c_i, c'_i \in K$, and $h \in K[x, y]$. If η is as in part 3 of Definition 4, then $\eta \le \sum_{i=k+1}^{k+\ell}(b_i + b'_i)$.

Proof. Suppose that $\mathfrak{a}$ is proper ∂-finite. Let p be an irreducible factor of u such that both $\deg_x(p)$ and $\deg_y(p)$ are nonzero. We have $(u; r)_x \mid (u; r+1)_x$ for all $r \ge 0$. By the condition in part 1 of Def. 4, the set of irreducible factors of the left borders of $(u; r)_x$ for all $r \ge 0$ is finite. Therefore, there is at least one positive integer s such that $\sigma_x^s(p) = \sigma_y^t(p)$ for some $t \in \mathbb{Z}$. By Corollary 1 in [2, page 400], p is integer-linear.

Conversely, if u is a product of integer-linear polynomials $a_i x + b_i y + c_i$, then it is sufficient to prove the more specific claim, because if u and v are not given in this form, we can multiply both u and U with the missing factors such as to complete the rising and falling factorials.

In order to keep the notation simple, let us only discuss the factors $(a_i x + b_i y + c_i)$. An analogous argument applies to the other factors $(a'_i x - b'_i y + c'_i)$.

Let $r \ge 1$. For fixed i, write $p_i = a_i x + b_i y + c_i$ and let $s, t \in \mathbb{N}$ be such that $a_i r = s b_i + t$. Then

$$\begin{aligned}
&((a_i x + b_i y + c_i)^{\overline{a_i}}; r)_x = p_i^{\overline{a_i r}} \\
&= (p_i)(p_i+1)\cdots\cdots(p_i + b_i - 1) \\
&\quad \times (p_i + b_i)(p_i + b_i + 1)\cdots(p_i + 2b_i - 1) \\
&\quad \vdots \\
&\quad \times (p_i + (s-1)b_i)(p_i + (s-1)b_i + 1)\cdots(p_i + sb_i - 1) \\
&\quad \times (p_i + sb_i)\cdots(p_i + sb_i + t) \\
&= \prod_{j=0}^{t} (p_i + j; s+1)_y \prod_{j=t+1}^{b_i - 1} (p_i + j; s)_y.
\end{aligned}$$

Therefore, if we choose $f_1, \ldots, f_m$ to be all the linear factors $p_i + j$ $(i = 1, \ldots, k;\ j = 0, \ldots, b_i - 1)$ and set

$$g = \prod_{i=k+1}^{k+\ell} p_i^{\overline{a_i r}},$$

then we will have $u = g \prod_{i=1}^{m} (f_i; s_i)_y$ for certain $s_i \in \mathbb{N}$ with $s_i \ge \lfloor a_i r / b_i \rfloor$ and

$$g\lceil_y = \prod_{i=k+1}^{k+\ell} p_i^{\overline{b_i}},$$

the y-degree of which is $\sum_{i=k+1}^{k+\ell} b_i$, as claimed. ∎

Example 10. *For fixed $n \ge 0$ and ϱ, the annihilator $\mathfrak{a}$ of the function*

$$f(x, y) = \frac{1 + 2^y + 3^y + \cdots + n^y}{\Gamma(x + \varrho y)}$$

is proper ∂-finite with $\eta = 0$, dimension n, and height ϱ. As the exponential terms k^y $(k = 1, \ldots, n)$ are algebraically independent over $K(x, y)$, there is no nontrivial $W \in \mathbb{A}/\mathfrak{a}$ for which $\partial_y W = 0$. Therefore $\phi = 0$. The minimal telescoper for $f(x, y)$ is

$$T = (\partial_x^\varrho - 1)(\partial_x^\varrho - 2)\cdots(\partial_x^\varrho - n)$$

and its order $n\varrho = n\varrho + \phi$ matches the bound of Theorem 6. The corresponding certificate C cannot be written in such a nice form and is therefore not displayed here.

For the hypergeometric case, our bound does not exactly reduce to the known bounds stated in Section 2.4 for this case. Our bound is at the same time better and worse than the old bound. It is worse because for the hypergeometric case it turns out that because of an additional cancellation the term $\eta = \deg_y(g\lceil_y)$ does not contribute to the order. It is slightly better because we work in the Ore algebra where ∂_y represents the forward difference rather than the shift operator, and for certain hypergeometric terms, it turns out that this improves the bound by 1. For example, for the hypergeometric term $(x+3y+1)!/(x+3y+\sqrt{2})!$ our bound evaluates to 2, which is indeed the order of the minimal telescoper, while the bound of Section 2.4 only predicts a telescoper of order 3.

4.3 Mixed and Other Cases

Thanks to the generality in which we stated our results in Section 3 we can not only deal with the pure differential or pure shift cases discussed above, but also with mixed cases where the two indeterminates x and y are different in nature (discrete versus continuous). In these cases, a necessary condition for an ideal to be proper ∂-finite is that the polynomial u is split, i.e., that it can be written as $u(x, y) = u_1(x)u_2(y)$. A polynomial that violates this condition can never be y-proper. We now give an example where x is a continuous variable and y is discrete, corresponding to a definite sum over y for which a differential equation in x is sought.

Example 11. *Let $\mathbb{A} = K(x, y)[\partial_x, \partial_y]$ be the Ore algebra given by $\sigma_x = \mathrm{id}$, $\delta_x = \frac{\mathrm{d}}{\mathrm{d}x}$, $\sigma_y(y) = y + 1$, and $\delta_y = \sigma_y - \mathrm{id}$. With respect to this algebra each member of the family*

$$f_k(x, y) = (y + 1)^{-k} J_y(x), \quad k \in \mathbb{N},$$

involving the Bessel function of the first kind, is ∂-finite. For any fixed k, the annihilator $\mathfrak{a}$ of $f_k(x, y)$ is generated by two operators, one of which corresponds to the famous Bessel differential equation $x^2\partial_x^2 + x\partial_x + x^2 - y^2$, and we have $n = \dim_{K(x,y)}(\mathbb{A}/\mathfrak{a}) = 2$. As a basis for $\mathbb{A}/\mathfrak{a}$ we choose the two monomials 1 and ∂_x so that the multiplication matrices are

$$U = \begin{pmatrix} 0 & x^2 \\ y^2 - x^2 & -x \end{pmatrix}$$

$$V = \begin{pmatrix} xy(y+1)^k - x^2(y+2)^k & -x^2(y+1)^k \\ (y+1)^k(x^2 - y^2 - y) & x(y+1)^{k+1} - x^2(y+2)^k \end{pmatrix}$$

with denominators $u = x^2$ and $v = x^2(y + 2)^k$. Obviously u is y-proper and therefore the height of $\mathfrak{a}$ is (at most) $\max\{\deg_y(v) - 1, \deg_y(V)\} = k + 2$. Taking $\phi = 0$ into account, Theorem 6 produces the bound $2(k + 2)$ for the order of the telescoper. In contrast, the minimal telescoper conjecturally is of order $2k + 1$ (we verified this for $0 \le k \le 20$), so our bound overshoots by 3.

Last but not least let us emphasize that all our results also apply to the q-case, where $\sigma_y(y) = qy$ and $\delta_y = \sigma_y - \mathrm{id}$; it is very much analogous to the shift case.

5. CONCLUSION AND OPEN QUESTIONS

We have shown that the reasoning of Apagodu and Zeilberger applies in the general setting of ∂-finite ideals in Ore algebras.

As a sufficient condition for guaranteeing the existence of a telescoper, we have introduced the notion of "proper" ∂-finite ideals, in analogy with the notion of proper hypergeometric terms in classical summation theory. For hypergeometric terms, Wilf and Zeilberger conjectured in 1992 that they are proper if and only if they are holonomic. Slightly modified versions of this conjecture were proved independently [20, 2] for the shift case, and recently [9] for general hypergeometric terms. It is now tempting to conjecture that, more generally, a ∂-finite ideal is proper if and only if it is holonomic.

For the hypergeometric case, Abramov [1] pointed out that proper is only a sufficient condition, but it is not necessary for the existence of a telescoper, and he formulates a finer condition which is necessary and sufficient. Abramov's existence criterion has been extended to the q-shift case and mixed cases [10, 6]. It would be interesting to have an analogous result for the ∂-finite case.

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