ON TURÁN'S INEQUALITY FOR LEGENDRE POLYNOMIALS

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Abstract. Let

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x),$$

where P_n is the Legendre polynomial of degree n. A classical result of Turán states that $\Delta_n(x) \geq 0$ for $x \in [-1, 1]$ and $n = 1, 2, 3, ...$ Recently, Constantinescu improved this result. He established

$$\frac{h_n}{n(n+1)}(1 - x^2) \leq \Delta_n(x) \quad (-1 \leq x \leq 1;\ n = 1, 2, 3, ...),$$

where h_n denotes the n-th harmonic number. We present the following refinement. Let $n \geq 1$ be an integer. Then we have for all $x \in [-1, 1]$:

$$\alpha_n\,(1 - x^2) \leq \Delta_n(x)$$

with the best possible factor

$$\alpha_n = \mu_{[n/2]}\,\mu_{[(n+1)/2]}.$$

Here, $\mu_n = 2^{-2n}\binom{2n}{n}$ is the normalized binomial mid-coefficient.

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1. Introduction

The Legendre polynomial of degree n can be defined by

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n \quad (n=0,1,2,...),$$

which leads to the explicit representation

$$P_n(x) = \frac{1}{2^n}\sum_{\nu=0}^{[n/2]}(-1)^\nu \frac{(2n-2\nu)!}{\nu!(n-\nu)!(n-2\nu)!}x^{n-2\nu}.$$

(As usual, $[x]$ denotes the greatest integer not greater than x.) The most important properties of $P_n(x)$ are collected, for example, in [1] and [16]. Legendre polynomials belong to the class of Jacobi polynomials, which are studied in detail in [3] and [13]. These functions have various interesting applications. For instance, they play an important role in numerical integration; see [12].

The following beautiful inequality for Legendre polynomials is due to P. Turán [15]:

$$\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0 \quad \text{for } -1 \leq x \leq 1 \text{ and } n \geq 1.^3 \tag{1.1}$$

This inequality has found much attention and several mathematicians provided new proofs, far-reaching generalizations, and refinements of (1.1). We refer to [8, 11, 9, 14] and the references given therein.

In this paper we are concerned with a remarkable result published by E. Constantinescu [7] in 2005. He offered a new refinement and a converse of Turán's inequality. More precisely, he proved that the double-inequality

$$\frac{h_n}{n(n+1)}(1-x^2) \leq \Delta_n(x) \leq \frac{1}{2}(1-x^2) \tag{1.2}$$

is valid for $x \in [-1,1]$ and $n \geq 1$. Here, $h_n = 1+1/2+\cdots+1/n$ denotes the n-th harmonic number.

It is natural to ask whether the bounds given in (1.2) can be improved. In the next section, we determine the largest number α_n and the smallest number β_n such that we have for all $x \in [-1,1]$:

$$\alpha_n\,(1-x^2) \leq \Delta_n(x) \leq \beta_n\,(1-x^2).$$

We show that the right-hand side of (1.2) is sharp, but the left-hand side can be improved. It turns out that the best possible factor α_n can be expressed in terms of the normalized binomial mid-coefficient

$$\mu_n = 2^{-2n}\binom{2n}{n} = \frac{1\cdot 3\cdot 5\cdot ...\cdot(2n-1)}{2\cdot 4\cdot 6\cdot ...\cdot (2n)} \quad (n=0,1,2,\dots).$$

We remark that μ_n has been the subject of recent number theoretic research; see [2] and [5].

In our proof we reduce the desired refinement of Turán's inequality to another inequality, which also depends polynomially on Legendre polynomials. This latter inequality is amenable to a recent computer algebra procedure [10, 11]. The procedure sets up a formula that encodes the induction step of an inductive proof of the inequality and, replacing the quantities $P_n(x), P_{n+1}(x), \dots$ by real variables $Y_1, Y_2, \dots,$ transforms the induction step formula into a polynomial formula in finitely many variables. The recurrence relation of the Legendre polynomials translates into polynomial equations in the Y_k, which are added to the induction step formula. The truth of the resulting formula for all real $Y_1, Y_2, \dots$ can be decided algorithmically and is a sufficient (in general not necessary!) condition for the truth of the initial inequality, if we assume that sufficiently many initial values have been checked.

[3] A nice anecdote about Turán reveals that he used (1.1) as his 'visiting card'; see [4].

2. Main result

The following refinement of (1.2) is valid.

Theorem. *Let n be a natural number. For all real numbers $x \in [-1,1]$ we have*

$$\alpha_n\,(1-x^2) \le P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \le \beta_n\,(1-x^2) \tag{2.1}$$

with the best possible factors

$$\alpha_n = \mu_{[n/2]}\,\mu_{[(n+1)/2]} \quad \textit{and} \quad \beta_n = \frac{1}{2}. \tag{2.2}$$

Proof. We define for $x \in (-1,1)$ and $n \ge 1$:

$$f_n(x) = \frac{\Delta_n(x)}{1-x^2}.$$

We have $f_1(x) \equiv \alpha_1 = \beta_1 = 1/2$. First, we prove that f_n is strictly increasing on $(0,1)$ for $n \ge 2$. Differentiation yields

$$f_n'(x) = \frac{2x\Delta_n(x) + (1-x^2)\Delta_n'(x)}{(1-x^2)^2}.$$

Using the well-known formulas

$$P_n'(x) = \frac{n+1}{1-x^2}(xP_n(x) - P_{n+1}(x))$$

and

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

we obtain the representation

$$n(1-x^2)^2 f_n'(x) = (n-1)xP_n(x)^2 - (2nx^2+x^2-1)P_n(x)P_{n+1}(x) + (n+1)xP_{n+1}(x)^2. \tag{2.3}$$

We prove the positivity of the right-hand side of (2.3) on $(0,1)$ by typing

In[1]:= $\boldsymbol{<<}$ **SumCracker.m**

SumCracker Package by Manuel Kauers – © RISC Linz – V 0.3 2006-05-24

In[2]:= **ProveInequality[**

$$\begin{aligned}&\boldsymbol{((n-1)\,x\,\mathrm{LegendreP}[n,x]^2}\\ &\quad\boldsymbol{-\,(2nx^2+x^2-1)\mathrm{LegendreP}[n,x]\mathrm{LegendreP}[n+1,x]}\\ &\quad\boldsymbol{+\,(n+1)\,x\,\mathrm{LegendreP}[n+1,x]^2)>0,}\\ &\boldsymbol{\mathrm{From}\to 2, \mathrm{Using}\to\{0<x<1\}, \mathrm{Variable}\to n]}\end{aligned}$$

into Mathematica, obtaining, after a couple of seconds, the output

Out[2]= True

It follows from this that f_n is strictly increasing on $(0,1)$ for $n \ge 2$. Since

$$P_n(x) = (-1)^n P_n(-x),$$

we conclude that f_n is even. Thus, we obtain

$$f_n(0) < f_n(x) < f_n(1) \quad \text{for } -1 < x < 1,\ x \ne 0. \tag{2.4}$$

We have

$$P_n(1) = 1 \quad \text{and} \quad P_n'(1) = \frac{1}{2}n(n+1).$$

Therefore,

$$\Delta_n(1) = 0 \quad \text{and} \quad \Delta_n'(1) = -1.$$

Applying l'Hospital's rule gives

$$f_n(1) = \lim_{x\to 1} \frac{\Delta_n(x)}{1-x^2} = -\frac{1}{2}\Delta_n'(1) = \frac{1}{2}. \tag{2.5}$$

Since

$$P_{2k-1}(0) = 0 \quad \text{and} \quad P_{2k}(0) = (-1)^k \mu_k,$$

we get

$$f_{2k-1}(0) = \mu_{k-1}\mu_k \quad \text{and} \quad f_{2k}(0) = {\mu_k}^2. \tag{2.6}$$

Combining (2.4)–(2.6) we conclude that (2.1) holds with the best possible factors α_n and β_n given in (2.2). □

Remarks. (1) The proof of the Theorem reveals that for $n \geq 2$ the sign of equality holds on the left-hand side of (2.1) if and only if $x = -1, 0, 1$ and on the right-hand side if and only if $x = -1, 1$.

(2) The numbers $\mu_p\mu_q$ $(p, q = 0, 1, 2, \ldots; p \leq q)$ are the eigenvalues of Liouville's integral operator for the case of a planar circular disc of radius 1 lying in $\mathbf{R}^3$; see [6].

(3) The automated proving procedure can be applied to (2.1) directly. However, owing to the computational complexity of the method, we did not obtain any output after a reasonable amount of computation time.

(4) The Mathematica package SumCracker used in the proof of the Theorem contains an implementation of the proving procedure described in [10]. It is available online at

http://www.risc.uni-linz.ac.at/research/combinat/software

(5) The normalized Jacobi polynomial of degree n is defined for $\alpha, \beta > -1$ by

$$R_n^{(\alpha,\beta)}(x) = {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2).$$

The special case $\alpha = \beta$ leads to the normalized ultraspherical polynomial

$$R_n^{(\alpha,\alpha)}(x) = {}_2F_1(-n, n+2\alpha+1; \alpha+1; (1-x)/2) = \frac{(-1)^n}{2^n\,(\alpha+1)_n}\frac{1}{(1-x^2)^\alpha}\frac{d^n}{dx^n}(1-x^2)^{n+\alpha},$$

where $(a)_n$ denotes the Pochhammer symbol. Obviously, we have $R_n^{(0,0)}(x) = P_n(x)$. We conjecture that the following extension of our Theorem holds.

Conjecture. *Let $\alpha > -1/2$ and $n \geq 1$. For all $x \in [-1, 1]$ we have*

$$a_n^{(\alpha)}\,(1-x^2) \leq R_n^{(\alpha,\alpha)}(x)^2 - R_{n-1}^{(\alpha,\alpha)}(x)R_{n+1}^{(\alpha,\alpha)}(x) \leq b_n^{(\alpha)}\,(1-x^2)$$

with the best possible factors

$$a_n^{(\alpha)} = \mu_{[n/2]}^{(\alpha)}\,\mu_{[(n+1)/2]}^{(\alpha)} \quad \text{and} \quad b_n^{(\alpha)} = \frac{1}{2(\alpha+1)}.$$

Here, $\mu_n^{(\alpha)} = \mu_n / \binom{n+\alpha}{n}$.

(6) Gasper [9] has shown that the normalized Jacobi polynomials satisfy

$$R_n^{(\alpha,\beta)}(x)^2 - R_{n-1}^{(\alpha,\beta)}(x)R_{n+1}^{(\alpha,\beta)}(x) \geq 0 \quad (-1 \leq x \leq 1)$$

if and only if $\beta \geq \alpha > -1$. More general criteria for a family of orthogonal polynomials to satisfy a Turán-type inequality are given by Szwarc [14].

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