THE COMPLETE GENERATING FUNCTION FOR GESSEL WALKS IS ALGEBRAIC

— SUPPLEMENT TO SECTION 3.2 —

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ABSTRACT. We provide here a detailed proof of the fact that the two polynomials `polU`, `polV`, called P_1 and P_2 in Section 3.2 of the main article, admit unique power series solutions in $\mathbb{Q}[[x,t]]$ and $\mathbb{Q}[[y,t]]$, respectively.

1. A Simple Example First

For illustration of the argument, consider a univariate example first. Let

$$\begin{aligned}P(T,t) := {} & 512t^{10}T^6 + 768(t-1)t^8T^5 + 64(18t^2-12t+7)t^6T^4 + 64(t-1)(13t^2-2t+2)t^4T^3 \\ & + 2(288t^4-264t^3+136t^2-16t+9)t^2T^2 + (t-1)(192t^4-72t^3+32t^2+1)T \\ & + (118t^4-72t^3+16t^2-t+1).\end{aligned}$$

We want to show that there is a power series $f(t) \in \mathbb{Q}[[t]]$ with $P(f(t),t) = 0$. (This is the Kreweras example with $x = 2, y = 0$.)

By Puiseux's theorem, there exists a full system of solutions in the ring

$$\mathbb{Q}\{t\} := \bigcup_{r\in\mathbb{N}} \bigcup_{\alpha\in\mathbb{Q}} t^\alpha \mathbb{Q}[[t^{1/r}]].$$

Let us write the elements of this ring in the form

$$f(t) = \sum_{q\in\mathbb{Q}} c_q t^q$$

that saves us from making α or r explicit. We understand this notation so that the set of $q \in \mathbb{Q}$ for which $c_q \neq 0$ is such that $f(t)$ indeed belongs to $\mathbb{Q}\{t\}$. In particular, we will always have $c_q = 0$ for all sufficiently small q.

The ring $\mathbb{Q}\{t\}$ is a differential ring, and we can find with gfun (or otherwise) a differential operator $L \in \mathbb{Q}(t)\langle D_t\rangle$ such that every $f \in \mathbb{Q}\{t\}$ with $P(f,t) = 0$ also satisfies $L \cdot f = 0$. For the present example, such an operator is given by

$$\begin{aligned}L = {} & 2t^3(3t-1)(9t^2+3t+1)(31t^2+4t-4)(585t^4-36t^3-532t^2+144t-64)D_t^4 \\ & + t^2(15668640t^9+681372t^8-17532936t^7+2475135t^6-149868t^5-198792t^4+122560t^3 \\ & + 11504t^2+17600t-5376)D_t^3 + 3t(23502960t^9+457812t^8-28669824t^7+5360739t^6 \\ & - 1540887t^5-372544t^4+144748t^3+54384t^2+15232t-4608)D_t^2 + 6(15668640t^9 \\ & - 70956t^8-21136104t^7+4870284t^6-2066280t^5-251411t^4+59814t^3+51336t^2+4288t \\ & - 1280)D_t + 24t(979290t^7-27945t^6-1475100t^5+398520t^4-208200t^3-11136t^2 \\ & - 384t+3584).\end{aligned}$$

This differential operator translates into a difference operator $R \in \mathbb{Q}[q]\langle S_q\rangle$ which annihilates the coefficients. In the example,

$$\begin{aligned} R = & -256(q+9)(q+10)(q+11)(2q+21)S_q^9 + 64(q+8)(q+9)(q+10)(26q+249)S_q^8 \\ & -16(q+9)(90q^3+451q^2-11088q-71533)S_q^7 + 4(2216q^4+70528q^3+831865q^2 \\ & +4309124q+8267412)S_q^6 - 2(3488q^4+148228q^3+1999216q^2+10992341q+21463437)S_q^5 \\ & +9(4048q^4+23828q^3-521817q^2-5203475q-12531324)S_q^4 + 9(q+4)(10850q^3 \\ & +296715q^2+2369493q+5793714)S_q^3 - 216(q+3)(q+4)(4744q^2+57451q+177876)S_q^2 \\ & +972(q+2)(q+3)(q+4)(68q-47)S_q + 979290(q+1)(q+2)(q+3)(q+4). \end{aligned}$$

This operator translates into a recurrence equation

$$\begin{aligned} 256q(q+1)(q+2)(2q+3)c_q = {} & 64(q-1)q(q+1)(26q+15)c_{q-1} \\ & -16q(90q^3-1979q^2+2664q-820)c_{q-2} + \cdots \\ & +979290(q-8)(q-7)(q-6)(q-5)c_{q-9}. \end{aligned}$$

Whenever a series $f(t) = \sum_{q\in\mathbb{Q}} c_q t^q \in \mathbb{Q}\{t\}$ is such that $P(f(t),t)=0$, then its coefficients must satisfy this recurrence. Because of the recurrence, a particular coefficient c_q can be nonzero only for two reasons: (i) at least one of $c_{q-1},\dots,c_{q-9}$ is also nonzero, or (ii) $q(q+1)(q+2)(2q+3)=0$. Since $c_q=0$ for all sufficiently small q, it follows that the support of $f(t)$ must be a subset of

$$\mathbb{N} \cup (-1+\mathbb{N}) \cup (-2+\mathbb{N}) \cup (-\tfrac{3}{2}+\mathbb{N}) = (-2+\mathbb{N}) \cup (-\tfrac{3}{2}+\mathbb{N}).$$

In particular, if $c_n \neq 0$ for some $n\in\mathbb{Z}$, then $c_0\neq 0$ or $c_{-1}\neq 0$ or $c_{-2}\neq 0$, and, more importantly, if $c_{n+1/2}\neq 0$ for some $n\in\mathbb{Z}$, then $c_{-3/2}\neq 0$.

Applying now Puiseux' algorithm to the original polynomial $P(T,t)$, we find that there exists a solution starting as

$$f(t) = 1 + 2t^2 + 2t^3 + 8t^4 + \mathrm{O}(t^5).$$

This must be a power series, for if that series involved some nontrivial term $c_q t^q$ with $q=u/v\geq 5$, $\gcd(u,v)=1$, $v\neq 1$, then by the above reasoning, $v=2$ and there would appear a nontrivial term $c_{-3/2}t^{-3/2}$ in $f(t)$, which is not the case.

2. Existence of a Solution of P_1

Our actual problems are technically slightly more involved, because they have one variable more. However, the basic idea of the construction will be the same. Let us first consider the case $U(t,x)$. Let $P_1(t,x,T)$ be the guessed minimal polynomial of $U(t,x)$. We show that there exists a unique power series $U_{\mathrm{cand}}(t,x)$ with $P_1(t,x,U_{\mathrm{cand}}(t,x))=0$.

Consider solutions $f(t,x)$ of the form

$$f(t,x) = \sum_{p,q\in\mathbb{Q}} c_{p,q} t^p x^q$$

with $c_{p,q}\in\mathbb{Q}$, nonzero only for some appropriate set of indices. The condition in the univariate case that the coefficients are identically zero for all indices q below some bound u translates in the bivariate case into the condition that they be identically zero for all indices (p,q) outside of some translate of a certain halfplane

$$up + vq \leq 0.$$

The coefficients u,v defining this halfplane are not entirely up to our choice, but they depend on the Newton polytope of P. According to Theorem 3.6 in [2], we can choose any (u,v) such that u and v are linearly independent over $\mathbb{Q}$ and (u,v) belongs to the "normal cone" $C^*(e)$ of some "admissible edge" e in the Newton polytope of Q (Notions as in [2]). In our case, we can choose the edge $e=(44,32,24)-(4,12,4)$. Then, continuing to use notions defined in [2], the barrier wedge is

$$W(e) = \{\, (x,y,z)\in\mathbb{R}^3 : z \leq 12 + x - y \wedge 3z \leq 4 + x + y \,\},$$

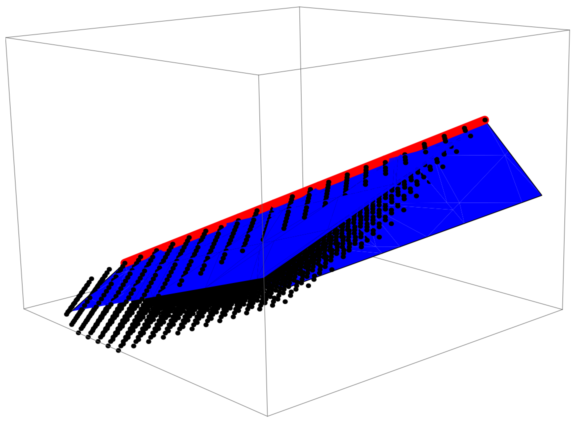
the barrier cone is

$$C(e) = \{\, (x, y) \in \mathbb{R}^2 : x + y \geq 0 \wedge x - y \geq 0 \,\},$$

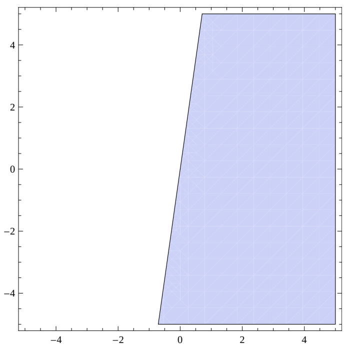
and the normal cone is

$$C^*(e) = \{\, (x, y) \in \mathbb{R}^2 : x \leq 0 \wedge x \leq y \leq -x \,\}.$$

Consequently, we can choose for instance $(u, v) = (-1, \frac{1}{10}\sqrt{2}) \in C^*(e)$. In the figure below, the black points form the support of Q, our edge e is drawn in red, and the two blue half planes form the boundary of $W(e)$.



Our choice of (u, v) was made such as to ensure the existence of a series solution whose support belongs to (some translate of) the half plane $up + vq \leq 0$ shown here:



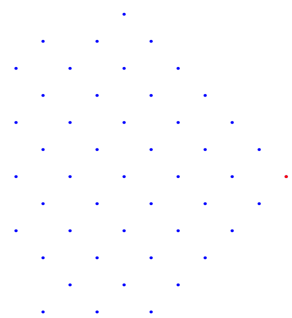
Next, we can find a system of differential equations $L_1, L_2, \ldots, L_5 \in \mathbb{Q}[t, x]\langle D_t, D_x\rangle$ such that $P_1(f(t, x), t, x) = 0$ implies $L_i \cdot f = 0$ $(i = 1, \ldots, 5)$. These operators are posted on our website [1]. The differential operators give rise to difference operators $R_1, R_2, \ldots, R_5 \in \mathbb{Q}[p, q]\langle S_p, S_q\rangle$ such that $R_i \cdot c_{p,q} = 0$ for any set of coefficients belonging to a solution f. These operators translate to multivariate recurrence equations of the form

$$\begin{aligned}
a_1(p, q)c_{p,q} &= \ldots c_{p+i,q+j} \ldots \\
a_2(p, q)c_{p,q} &= \ldots c_{p+i,q+j} \ldots \\
a_3(p, q)c_{p,q} &= \ldots c_{p+i,q+j} \ldots \\
a_4(p, q)c_{p,q} &= \ldots c_{p+i,q+j} \ldots \\
a_5(p, q)c_{p,q} &= \ldots c_{p+i,q+j} \ldots
\end{aligned}$$

where

$$\begin{aligned}
a_1(p,q) &= 2(9p^2q + 15p^2 - 9pq^3 - 63pq^2 - 74pq + 18q^4 + 63q^3 + 25q^2 - 51q - 15),\\
a_2(p,q) &= 2(p+1)(3p+4)(-p + 3q^2 + 6q + 1),\\
a_3(p,q) &= -2(9p^3q + 9p^3 + 36p^2q + 57p^2 - 126pq^2 - 205pq - 9p + 63q^4 + 252q^3\\
&\qquad + 182q^2 - 120q - 57)\\
a_4(p,q) &= 2(9p^4 + 54p^3 + 182p^2 - 378pq^2 - 756pq - 54p + 189q^4 + 756q^3 + 546q^2 - 420q - 191)\\
a_5(p,q) &= 2(-15p^2q - 30p^2 + 90pq^2 + 140pq + 9q^5 - 100q^3 - 70q^2 + 91q + 30),
\end{aligned}$$

and the right hand sides are linear combinations of certain shifted versions of $c_{p,q}$. The full recurrences are available on our website. The shifts appearing in the first recurrence are given by the picture below. The red bullet indicates the point (p,q) and a blue bullet at point $(p+i,q+j)$ represents a term $c_{p+i,q+j}$ on the right hand side of the first recurrence. The other recurrences look similar.



In a solution $f(t,x)$ whose coefficients are nonzero in a half plane as chosen before, a coefficient $c_{p,q}$ can cease to be zero only for two reasons: (i) one of the coefficients $c_{p+i,q+j}$ appearing on the right hand side of one of the recurrences is already nonzero, or (ii) $a_1(p,q) = a_2(p,q) = a_3(p,q) = a_4(p,q) = a_5(p,q) = 0$. The latter condition is satisfied precisely for

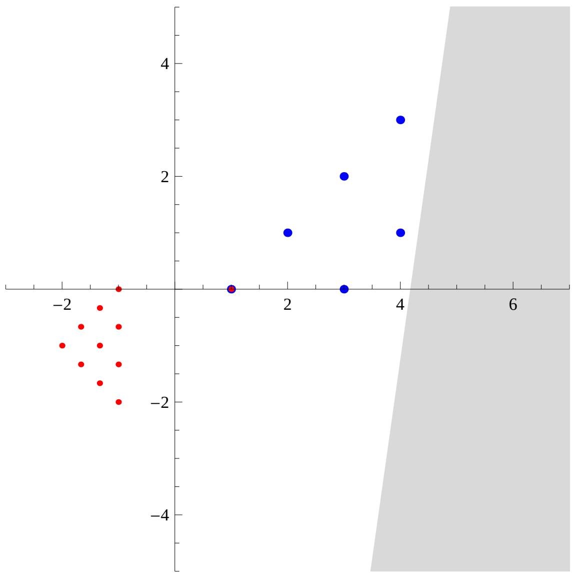
$$\begin{aligned}
(p,q) \in \bigl\{&(1,0), (-1,0), (-\tfrac{4}{3},-1), (-2,-1), (-1,-\tfrac{2}{3}),\\
&(-\tfrac{5}{3},-\tfrac{2}{3}), (-1,-\tfrac{4}{3}), (-\tfrac{5}{3},-\tfrac{4}{3}), (-\tfrac{4}{3},-\tfrac{5}{3}), (-1,-2), (-\tfrac{4}{3},-\tfrac{1}{3})\bigr\}.
\end{aligned}$$

The support of any solution f whose coefficients are zero in a half plane as chosen before is therefore contained in a union of certain cones $v + C$ for $C \subseteq \mathbb{Z}^2$ lying entirely in the opposite half plane and v being one of the critical points above. Moreover, since all the shift distances i, j in our recurrence equations are integers, such f can only have some fractional exponents if it has a nonzero coefficient at $t^{-4/3}x^{-1}$ or at $t^{-1}x^{-2/3}$ or at $t^{-5/3}x^{-2/3}$ or at $t^{-1}x^{-4/3}$ or at $t^{-5/3}x^{-4/3}$ or at $t^{-4/3}x^{-5/3}$ or at $t^{-4/3}x^{-1/3}$.

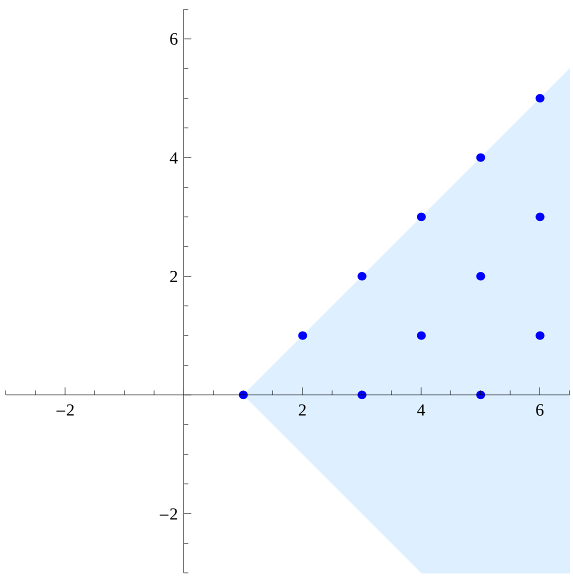
Applying now the generalized Puiseux algorithm of [2] to $P_1(T,t,x)$ in order to find the first terms of all the solutions f whose support lies in a half plane as chosen above, we find that there exists a solution starting as

$$f(t,x) = t + x + (5 + x^2)t^3 + (9x + x^3)t^4 + \text{further terms}$$

This series must belong to $\mathbb{Q}((x))[[t]]$, because the algorithm has been carried out to a point where all the "further terms" are guaranteed to belong to the gray region in the figure below. If f had a term with fractional exponent in the gray region, it would have as well a term with fractional exponent at one of the critical points (red in the figure), and in that case, they would have shown up already.



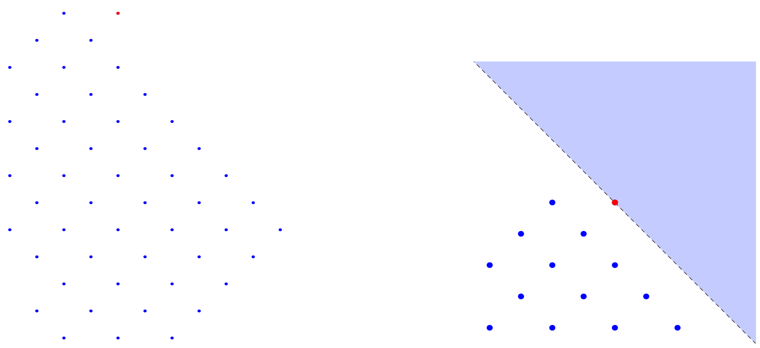
More precisely, the solution $f(t,x)$ only involves terms $c_{p,q}t^p x^q$ for $(p,q) \in \mathbb{Z}^2$ with $p \geq 1$ and $2-p \leq q \leq p$:



It remains to show that there are no terms with $q < 0$. To this end, we construct appropriate witness recurrences. Let $R_1, \dots, R_5$ be the difference operators corresponding to the recurrences recorded above (that came from the differential operators obtained from the polynomial equation). Let $M \in \mathbb{Q}[p,q]\langle S_p, S_q\rangle^{7\times 5}$ be the operator matrix posted on our website. and let $R'_1, \dots, R'_7$ be the operators obtained via

$$\begin{pmatrix} R'_1 \\ R'_2 \\ \vdots \\ R'_7 \end{pmatrix} := M \begin{pmatrix} R_1 \\ \vdots \\ R_5 \end{pmatrix}.$$

Clearly, the coefficients $c_{p,q}$ of any solution to our equation must also be annihilated by $R'_1, \dots, R'_7$. The matrix M was chosen such that the support of the R'_i looks as follows:



They have an isolated leading term of maximum total degree. If our solution $f(t,x)$ has some term $c_{p,q}t^p x^q$ with nonzero coefficient for some $q < 0$, then there will be a left-most line with slope -1 which contains such a term. This term must have an index (p,q) where all the leading coefficients of the above recurrences vanish simultaneously, because the support of these recurrence equations is such that all the $c_{p+i,q+j}$ occuring on its right hand side (blue bullets in the picture) are zero in order for the $c_{p,q}$ to be the left-most nonzero term. The leading terms of the recurrence equations, however, vanish simultanously only for p,q with

$$(q-2)(q-1)q(p+q+1)(p+q+3)(3p+3q+5)(3p+3q+7)=0$$

which is outside of the critical area, or for some finitely many additional points with non-integral or negative p-coordinates. Therefore, no non-trivial term $c_{p,q}t^p x^q$ with negative q can exist.

This completes the proof that $f(t,x)$ belongs to $\mathbb{Q}[x][[t]]$. According to McDonald's algorithm, the initial terms of all the other solutions involve fractional exponents. Therefore $f(t,x)$ is unique.

3. Existence of a Solution of P_2

Consider now the case $V(t,y)$. Let $P_2(T,t,y)$ be the guessed minimal polynomial for this series. This polynomial decomposes as

$$P_2(T,t,y) = Q(t,y,t^2T^2(1+y)^2 - T(3+y))$$

for some $Q \in \mathbb{Q}[T,t,y]$. We show first that $Q(t,t,y)$ has a power series solution $f(t,y)$, and then lift this result to the existence of a power series solution $g(t,y)$.

The reasoning is mostly the same as before. We just give the corresponding data.

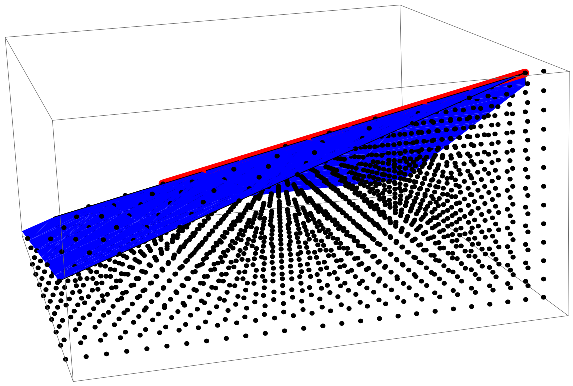
- We choose the edge $e = (22,24,12)–(6,8,4)$ (red in the figure below) and obtain the barrier wedge

$$W(e) = \{\,(x,y,z)\in\mathbb{R}^3 : -2 \le x+2y-6z \wedge -2 \le x-2z\,\}$$

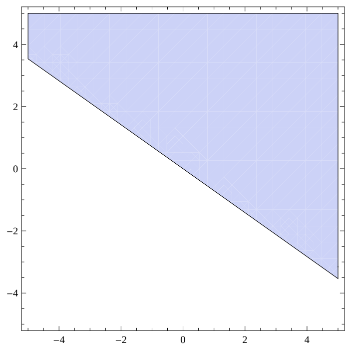
(its boundary half planes are blue in the figure below). The resulting barrier cone and normal cone are

$$C(e) = \{\,(x,y)\in\mathbb{R}^3 : 0 \le x+2y \wedge x \ge 0\,\},$$
$$C^*(e) = \{\,(x,y)\in\mathbb{R}^3 : x \le 0 \wedge 2x \le y \le 0\,\},$$

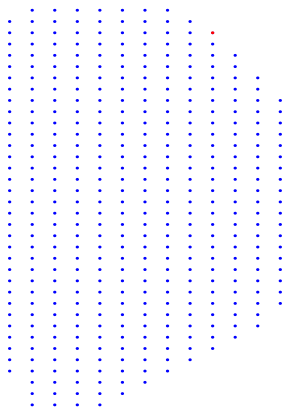
respectively.



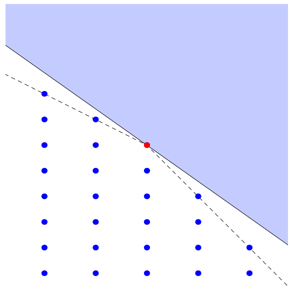
- From the normal cone $C^*(e)$, we choose $(u,v) = (-1, -\sqrt{2})$, defining the half plane depicted below. Observe that the boundary line has a slope of nearly -0.70.



- Five differential operators $L \in \mathbb{Q}[t,y]\langle D_t, D_y\rangle$ which annihilate any series solution of our polynomial equation are posted on the website.
- The recurrences corresponding to these differential operators are posted on the website. Call them $R_1, \ldots, R_5$. These operators involve shifts $c_{p+i,q+j}$ for (i,j) according to the following picture (red bullet = left hand side, blue bullets = right hand side):



Observe that the support is compatible with the half plane we have selected before:

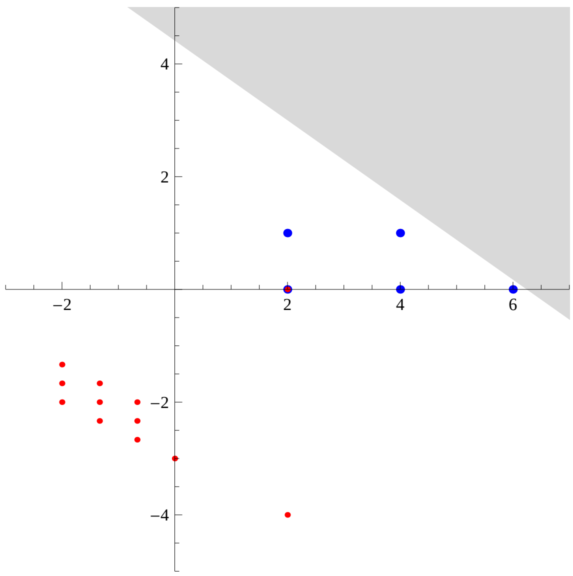


- The points where all the leading coefficients of all the recurrences in the recurrence system vanish are

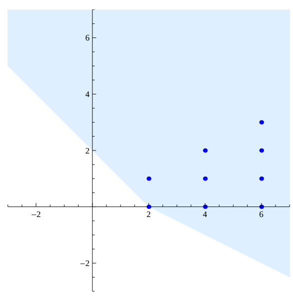
$$(-\tfrac{4}{3},-2),\quad (-\tfrac{2}{3},-2),\quad (-2,-2),\quad (2,0),\quad (2,-4),\quad (0,-3),\quad (-\tfrac{2}{3},-\tfrac{8}{3}),$$
$$(-2,-\tfrac{4}{3}),\quad (-\tfrac{2}{3},-\tfrac{7}{3}),\quad (-\tfrac{4}{3},-\tfrac{7}{3}),\quad (-2,-\tfrac{5}{3}),\quad (-\tfrac{4}{3},-\tfrac{5}{3}).$$

- There exists a solution starting like

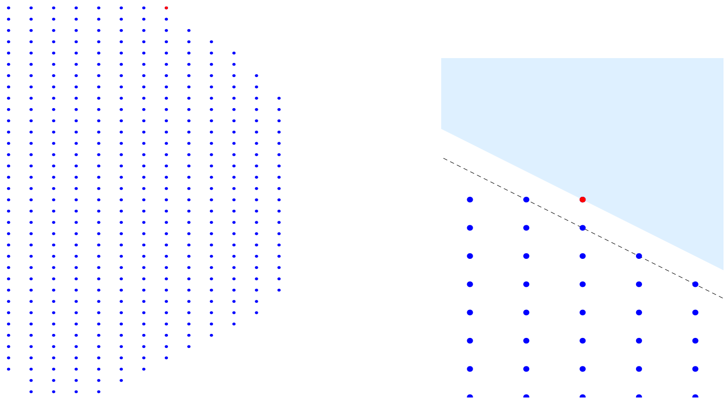
$$f(t,y)=-(y+3)t^2-2(y^2+7y+12)t^4+(-5y^3-44y^2-161y-218)t^6+\text{further terms}$$



We can conclude that there exists a solution $f(t,y)$ which has nonzero terms $c_{p,q}t^py^q$ only for $p,q\in\mathbb{Z}$ with $q\geq 2-p$ and $q\geq 1-p/2$.

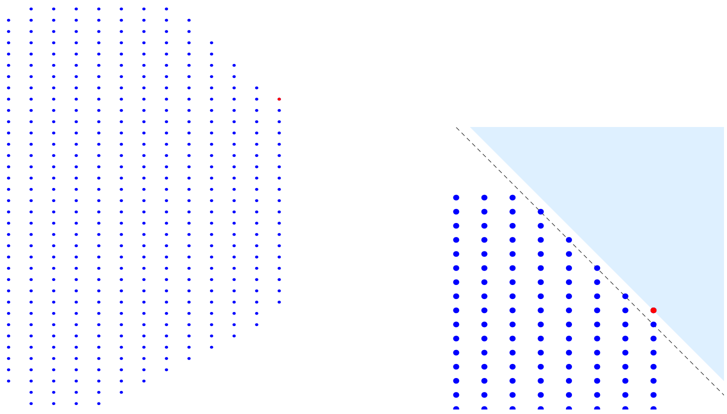


- This solution has no terms in the $(+,-)$ quadrant. For, multiplying the operator matrix posted on our website to the operator vector $(R_1,\dots,R_5)$ gives an array of six new operators, also posted on the website, whose support looks as follows.



If there was some coefficient with negative q then there would be a left-most line with slope $-\frac{1}{2}$ containing such a coefficient. By the shape of the recurrences above, such coefficients can only appear at indices (p, q) where all the leading coefficients of these recurrence equations vanish simultaneously. All these points, however, have irrational coordinates or lie outside the critical area.

- There are also no terms in the $(-, +)$ quadrant. For, multiplying a second operator matrix, posted on our website, to the operator vector $(R_1, \ldots, R_5)$ gives an array of two new operators (posted on the website) whose support looks as follows.



If there was some coefficient with negative p then there would exist a lowest line with slope -1 containing such a coefficient. By the shape of the recurrences above, such coefficients can only appear at indices (p, q) where the leading coefficients of these operators vanish simultaneously. This is the case on the algebraic curve

$$\begin{aligned}-18900p^3 + 40797p^2q^2 + 262989p^2q + 437004p^2 + 71784pq^3 + 680238pq^2 + 2081878pq \\ + 2021080p + 44586q^4 + 581004q^3 + 2789486q^2 + 5822204q + 4427280 = 0\end{aligned}$$

which lies outside of the ciritical area (because, according to CAD, the defining polynomial is positive there), and several isolated points all of which are outside the area or have irrational coordinates.

- Conclusion: There is a power series solution for $Q(t, y, f(t, y)) = 0$.
- Now consider the equation

$$Q(t, y, t^2g(t, y)^2(1 + y)^2 - g(t, y)(3 + y)) = 0.$$

The implicit function theorem applies here directly:

$$t^2T^2(1 + y)^2 - T(3 + y) - f(t, y)\Big|_{T,t,y\to 0} = 0$$

$$\frac{d}{dT}\Big(t^2T^2(1 + y)^2 - T(3 + y) - f(t, y)\Big)\Big|_{T,t,y\to 0} = -3 \neq 0$$

- Conclusion: There is a power series solution for $P(t, y, g(t, y)) = 0$. According to McDonald's algorithm, the initial terms of all the other solutions involve fractional exponents. Therefore $g(t, x)$ is unique.

References

[1] Alin Bostan and Manuel Kauers. The full counting function for Gessel walks is algebraic, 2008. (in preparation).
[2] John McDonald. Fiber polytopes and fractional power series. *Journal of Pure and Applied Algebra*, 104:213–233, 1995.

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