An Automated Zero Equivalence Test

- Some classes of sequences:
  1. $f(n+3) = f(n+1) - 3f(n)$: C-finite sequences (trivial)
  2. $f(n+2) = (n+3)f(n+1) - 3nf(n)$: P-finite sequences (routine)
  3. $f(n+2) = H(n+3)f(n+1) - 3^n f(n)$: $\Pi\Sigma$-sequences (Carsten Schneider)
  4. $f(n+2) = H(n+3)f(n+1)^2 - 3^n f(n)$: nonlinear recurrences (MK)
- The larger the class, the less can be done automatically with it.
- Consider the class $\mathcal{C}$, defined by *structural induction:*
  1. Constant sequences belong to $\mathcal{C}$
  2. if p is a polynomial in $f(n)$, $f(n+1), \dots, f(n+r-1)$ and $f_i(n+j)$ ($i = 1, \dots, s$, $j = 0, \dots, r$, $f_i \in \mathcal{C}$), then sequences defined by $f(n+r) = p$ or $f(n+r) = 1/p$ also belong to $\mathcal{C}$.
- Examples:
  1. all P-finite sequences
  2. $\mathrm{Fib}(n), 2^{\mathrm{Fib}(n)}, \mathrm{Fib}(2^n) \in \mathcal{C}$ (why?)
  3. $f(n) \in \mathcal{C} \implies \sum_{k=1}^{n} f(k), \prod_{k=1}^{n} f(k) \in \mathcal{C}$
  4. continued fractions, orthogonal polynomials, Sylvester sequences, etc.
- We have an algorithm which decides if a given element of $\mathcal{C}$ is equivalent to zero. (By $A = B \iff A - B = 0$ this allows to prove identities, note $A, B \in \mathcal{C} \implies A - B \in \mathcal{C}$)
- Idea: Use polynomial ideal arithmetic.
  1. Consider $f(n), f(n+1), \dots$ as variables in a polynomial ring.
  2. Consider the polynomial ideal I generated by the recurrences.
  3. Decide $f(n+r) \in I + \langle f(n), \dots, f(n+r-1) \rangle$ (e.g., by Gröbner basis techniques, see Winkler's course.)

     If yes: induction proof, otherwise increase r and try again.
- Example: Prove $\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n$
  1. Let $f_1(n) = n$, $f_2(n) = \sum_{k=1}^{n} f_1(k)$, $f_3(n) = f_2(n) - \frac{1}{2}f_1(n)^2 - \frac{1}{2}f_1(n)$.
  2. Consider

$$\begin{aligned} I = \big\langle & f_1(n+1) - f_1(n) - 1, \\ & f_2(n+1) - f_2(n) - f_1(n+1), \\ & f_3(n) - f_2(n) + \tfrac{1}{2}f_1(n)^2 + \tfrac{1}{2}f_1(n), \\ & f_3(n+1) - f_2(n+1) + \tfrac{1}{2}f_1(n+1)^2 + \tfrac{1}{2}f_1(n+1) \big\rangle \end{aligned}$$

  3. We have $f_3(n+1) \in I + \langle f_3(n) \rangle$, therefore the claim holds iff it holds for $n = 1$. ∎