Manuel Kauers
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Analytic Combinatorics
Homework 8

Exercise 33 Let $(F_k(x))_{k\geq 0}$ be a sequence in $\mathbb{C}[\![x]\!]$. Prove: The following three statements are equivalent.

(i) $\lim\limits_{k\to\infty} \operatorname{ord} F_k(x) = \infty$

(ii) $\sum\limits_{k=0}^{\infty} F_k(x)$ converges

(iii) $\prod\limits_{k=0}^{\infty} (1 + F_k(x))$ converges

(i) $\Rightarrow$ (ii) Let $m \in \mathbb{N}_0$. By (i), there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{ord} F_n(x) > m$ for all $n > n_0$. It follows that

$$\langle x^m \rangle \sum_{k=0}^{n} F_k(x) = \langle x^m \rangle \sum_{k=0}^{n_0} F_k(x)$$

for all $n > n_0$, as required.

(ii) $\Rightarrow$ (i) Let $m \in \mathbb{N}_0$. For every $m' \leq m$, we have by (ii) the existence of an $n_0(m') \in \mathbb{N}$ such that for all $n > n_0(m')$,

$$\langle x^{m'} \rangle \sum_{k=0}^{n} F_k(x) = \langle x^{m'} \rangle \sum_{k=0}^{n_0(m')} F_k(x),$$

therefore we have $\operatorname{ord} F_n(x) \neq m'$ for all $n > n_0(m')$. It follows that $\operatorname{ord} F_n(x) > m$ if $n > \max\{n_0(0), \ldots, n_0(m)\} =: n_0$. Summarizing,

$$\forall\ m \in \mathbb{N}\ \exists\ n_0 \in \mathbb{N}\ \forall\ n > n_0 : \operatorname{ord} F_n(x) > m,$$

as we wished to prove.

(i) $\Rightarrow$ (iii) Let $m \in \mathbb{N}_0$. By (i), there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{ord} F_n(x) > m$ for all $n > n_0$. By

$$\langle x^m \rangle \prod_{k=0}^{n+1} (1 + F_k(x)) = \langle x^m \rangle \prod_{k=0}^{n} (1 + F_k(x)) + \underbrace{\langle x^m \rangle \prod_{k=0}^{n} (1 + F_k(x)) F_{n+1}(x)}_{=0}$$

we get inductively that

$$\langle x^m \rangle \prod_{k=0}^{n} F_k(x) = \langle x^m \rangle \prod_{k=0}^{n_0} F_k(x)$$

for all $n > n_0$.

(iii) $\Rightarrow$ (i) Let $m \in \mathbb{N}_0$. For every $m' \le m$, we have by (iii) the existence of an $n_0(m') \in \mathbb{N}$ such that for all $n > n_0(m')$,

$$\langle x^{m'} \rangle \prod_{k=0}^{n} (1 + F_k(x)) = \langle x^{m'} \rangle \prod_{k=0}^{n_0(m')} (1 + F_k(x)), \tag{1}$$

therefore we have $\operatorname{ord} F_n(x) \neq m'$ for all $n > n_0(m')$, for otherwise

$$\langle x^{m'} \rangle \prod_{k=0}^{n} (1 + F_k(x)) = \langle x^{m'} \rangle \prod_{k=0}^{n-1} (1 + F_k(x)) + \underbrace{\langle x^{m'} \rangle\, \underbrace{F_n(x)}_{\text{ord}=m'} \underbrace{\prod_{k=0}^{n-1} (1 + F_k(x))}_{\text{ord}=0}}_{\neq 0}$$

$$\neq \langle x^{m'} \rangle \prod_{k=0}^{n-1} (1 + F_k(x))$$

in contradiction to (1).

It follows again that $\operatorname{ord} F_n(x) > m$ for all $n > \max\{n_0(0), \ldots, n_0(m)\} =: n_0$. Summarizing, we get

$$\forall\, m \in \mathbb{N}\ \exists\, n_0 \in \mathbb{N}\ \forall\, n > n_0 : \operatorname{ord} F_n(x) > m,$$

as we wished to prove.

Exercise 34 Prove Euler's transform:

$$\langle x^n \rangle \frac{1}{1-x} u\Big(\frac{x}{x-1}\Big) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \langle x^k \rangle u(x)$$

for $u(x) \in \mathbb{C}[\![x]\!]$ and $n \in \mathbb{N}_0$.

First note that $\operatorname{ord} \frac{x}{x-1} > 0$. This allows the following computation, writing $u(x) = \sum\limits_{n=0}^{\infty} u_n x^n$:

$$\begin{aligned}\frac{1}{1-x} u\Big(\frac{x}{x-1}\Big) &= \sum_{n=0}^{\infty} x \sum_{n=0}^{\infty} u_n \Big(-x \sum_{k=0}^{\infty} x^k \Big)^n = \sum_{n=0}^{\infty} x \Big(\sum_{n=0}^{\infty} (-1)^n u_n x^n + c(x) \Big) \\ &= \sum_{n=0}^{\infty} x \sum_{n=0}^{\infty} (-1)^n u_n x^n + \frac{c(x)}{1-x} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k u_k\, x^n + \frac{c(x)}{1-x}\end{aligned}$$

with some $c(x)$ with $\operatorname{ord} c(x) > n$. By $\operatorname{ord} \frac{c(x)}{1-x} = \operatorname{ord} c(x) > n$ we get

$$\langle x^n \rangle \frac{1}{1-x} u\Big(\frac{x}{x-1}\Big) = \langle x^n \rangle \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k u_k\, x^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k u_k$$

as claimed.

Exercise 35 $R := (\mathbb{C}[\![x]\!], D)$ is a differential ring.

Let $f(x) = \sum\limits_{n=0}^{\infty} f_n x^n, g(x) = \sum\limits_{n=0}^{\infty} g_n x^n \in \mathbb{C}[\![x]\!]$. Then

$$\begin{aligned}
D(f(x)+g(x)) &= D\sum_{n=0}^{\infty}(f_n+g_n)x^n = \sum_{n=0}^{\infty}(f_{n+1}+g_{n+1})(n+1)x^n \\
&= \sum_{n=0}^{\infty} f_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} g_{n+1}(n+1)x^n = Df(x)+Dg(x) \\
D(f(x)\cdot g(x)) &= D\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} f_k g_{n-k} x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\binom{n+1}{k} f_k g_{n+1-k}(n+1)x^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\binom{n+1}{k}(n+1-k+k) f_k g_{n+1-k} x^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\binom{n+1}{k} k f_k g_{n+1-k} x^n \\
&\qquad + \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\binom{n+1}{k} f_k (n+1-k) g_{n+1-k} x^n \\
&= \sum_{n=0}^{\infty}(n+1)f_{n+1}x^n \sum_{n=0}^{\infty} g_n x^n + \sum_{n=0}^{\infty} f_n x^n \sum_{n=0}^{\infty}(n+1)g_{n+1}x^n \\
&= (Df(x))g(x) + f(x)Dg(x).
\end{aligned}$$

Exercise 36 Let $R = (R, D)$ be a differential ring.

a) $C_D(R)$ is a subring of R.

b) for all $u \in R$, $v \in R^\times$ we have $D\frac{u}{v} = \frac{(Du)v - uDv}{v^2}$.

Ad a) If $a, b \in C_D(R)$ then, by definition, $Da = Db = 0$, and consequently $D(a+b) = Da + Db = 0$ and $D(ab) = (Da)b + aDb = 0$.

Ad b) It suffices to show that $D\frac{1}{v} = -\frac{1}{v^2}$ for $v \in R^\times$. First, $1 \in C_D(R)$, because $D1 = D(1 \cdot 1) = (D1)1 + 1D1 = 2D1$. (We assume characteristic zero; divisibility would be sufficient.) Now, $1 = v/v$ implies

$$0 = D1 = D(\frac{1}{v}v) = (D\frac{1}{v})v + \frac{1}{v}Dv$$

which in turn implies the claim.