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Analytic Combinatorics

Homework 7

Exercise 30 Let $\varepsilon > 0$ and let f, g be two functions which are analytic for $|x| < \varepsilon$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \qquad (|x| < \varepsilon).$$

Prove the transfer principle: If $f(x) = g(x)$ $(|x| < \varepsilon)$ then $a_n = b_n$ $(n \in \mathbb{N}_0)$.

For $|x| < \varepsilon$ the convergence of the series of f and g is absolute, so we have

$$f(x) - g(x) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n - b_n) x^n$$

for those x. Assume the a_n and b_n do not agree for all n. Then there exists a minimal $N \in \mathbb{N}_0$ with $a_N \neq b_N$, and we can write

$$\sum_{n=0}^{\infty} (a_n - b_n) x^n = \sum_{n=N}^{\infty} (a_n - b_n) x^n = x^N \underbrace{\sum_{n=N}^{\infty} (a_n - b_n) x^{n-N}}_{=:c(x)}.$$

c is analytic because f and g are, and so c is in particular continuous. By $a_N \neq b_N$ we get $c(0) \neq 0$, and continuity implies

$$\exists\ M > 0, \delta > 0\ \forall\ |x| < \delta : |c(x)| > M.$$

It follows

$$0 = |f(x) - g(x)| = |x^N c(x)| > M|x|^N$$

for all x with $|x| < \min\{\varepsilon, \delta\} =: \tilde{\varepsilon}$. For $x = \frac{1}{2}\tilde{\varepsilon}$ we get $0 > M(\frac{1}{2}\tilde{\varepsilon})^N > 0$. This is the desired contradiction.

Exercise 31 Prove

$$\sum_{k=0}^{n} kk! = (n+1)! - 1$$

Obvious by $\Delta n! = (n+1)! - n! = (n+1-1)n! = nn!$.

Exercise 32 Let $f(x) \in \mathbb{C}[\![x]\!]$ with $f(0) = 0$ and $\mu, \lambda \in \mathbb{C}$. Prove

$$(1 + f(x))^\lambda (1 + f(x))^\mu = (1 + f(x))^{\lambda+\mu}$$

in $\mathbb{C}[\![x]\!]$.

It is known from analysis that the derivative obeys a binomial theorem of the form

$$(gh)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} g^{(n-k)}(x) h^{(k)}(x) \qquad (g, h \in C^\infty(\mathbb{C}), n \in \mathbb{N}_0). \tag{1}$$

Let $\lambda, \mu \in \mathbb{C}$ and consider $g(x) = x^\lambda$, $h(x) = x^\mu$. Then $g^{(n)}(x) = \lambda^{\underline{n}} x^{\lambda - n}$, $h^{(n)}(x) = \mu^{\underline{n}} x^{\mu - n}$, $(gh)^{(n)}(x) = (\lambda + \mu)^{\underline{n}} x^{\lambda + \mu - n}$. Plugging this into (1) and evaluating at $x = 1$ gives the binomial theorem for falling factorials,

$$(\lambda + \mu)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} \lambda^{\underline{n-k}} \mu^{\underline{k}} \qquad (\lambda, \mu \in \mathbb{C}, n \in \mathbb{N}_0). \tag{2}$$

Note further that, by the definition of the binomial coefficient, we have

$$\binom{\alpha}{n} = \frac{\alpha^{\underline{n}}}{n!} \qquad (\alpha \in \mathbb{C}, n \in \mathbb{N}_0). \tag{3}$$

Now:

$$\begin{aligned}
&(1 + f(x))^\lambda (1 + f(x))^\mu \overset{\text{Def.}}{=} \sum_{n=0}^{\infty} \binom{\lambda}{n} f(x)^n \sum_{n=0}^{\infty} \binom{\mu}{n} f(x)^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{\lambda}{k} \binom{\mu}{n-k} f(x)^n \overset{(3)}{=} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{n!} \frac{\lambda^{\underline{k}}}{k!} \frac{\mu^{\underline{n-k}}}{(n-k)!} f(x)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^{\underline{k}} \mu^{\underline{n-k}} f(x)^n \overset{(2)}{=} \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda + \mu)^{\underline{n}} f(x)^n \\
&\overset{(3)}{=} \sum_{n=0}^{\infty} \binom{\lambda + \mu}{n} f(x)^n \overset{\text{Def.}}{=} (1 + f(x))^{\lambda + \mu}.
\end{aligned}$$