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# Analytic Combinatorics

## Homework 6

**Exercise 18** Prove

$$\frac{1}{(1-x)^m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k \qquad (x \neq 1, m \in \mathbb{Z})$$

We will present two inductions. The first induction provides the proof for $m \geq 0$, the second for $m \leq 0$. They share the same induction base $m = 0$:

$$1 = \frac{1}{(1-x)^0} = \sum_{k=0}^{\infty} \binom{k-1}{k} x^k.$$

The binomial coefficient $\binom{k-1}{k}$ vanishes for all $k$ except for $k = 0$ where it evaluates to 1. This completes the induction base.

Now suppose the claim is true for some $m \in \mathbb{Z}$ and prove it for $m + 1$.

$$\begin{aligned}\frac{1}{(1-x)^{m+1}} &= \frac{1}{(1-x)^m}\frac{1}{1-x} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k \sum_{k=0}^{\infty} x^k \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{k} \binom{m+i-1}{i} x^i x^{k-i} = \sum_{k=0}^{\infty} \Bigl(\sum_{i=0}^{k} \binom{m+i-1}{i}\Bigr) x^k\end{aligned}$$

It remains to show the identity

$$\sum_{i=0}^{k} \binom{m+i-1}{i} = \binom{m+k}{k}$$

which can be done, e.g., by induction on $k$. The case $k = 0$ is evident. As for the induction step, we have

$$\begin{aligned}\sum_{i=0}^{k+1} \binom{m+i-1}{i} &= \sum_{i=0}^{k} \binom{m+i-1}{i} + \binom{m+k}{k+1} = \binom{m+k}{k} + \binom{m+k}{k+1} \\ &= \binom{m+k+1}{k+1}.\end{aligned}$$

This completes the induction "to the right."

Now suppose the claim is true for some $m \in \mathbb{Z}$ and prove it for $m - 1$.

$$\begin{aligned}\frac{1}{(1-x)^{m-1}} &= \frac{1}{(1-x)^m}(1-x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k (1-x) \\ &= \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k - \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^{k+1}\end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{k=1}^{\infty} \binom{m+k-1}{k} x^k - \sum_{k=1}^{\infty} \binom{m+k-2}{k-1} x^k \\
&= 1 + \sum_{k=1}^{\infty} \left( \binom{m+k-1}{k} - \binom{m+k-2}{k-1} \right) x^k \\
&\overset{(*)}{=} 1 + \sum_{k=1}^{\infty} \left( (-1)^k \binom{-m}{k} + (-1)^k \binom{-m}{k-1} \right) x^k \\
&= 1 + \sum_{k=1}^{\infty} (-1)^k \binom{-m+1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{-m+1}{k} x^k \\
&\overset{(*)}{=} \sum_{k=0}^{\infty} \binom{k+m-2}{k} x^k.
\end{aligned}
$$

In $(*)$ we make use of the reflection formula to be proven in the next exercise. (The proof given there won't make use of the results presented here.)

The induction "to the left" is complete, and hence we are done.

*Remark:* The induction steps did not make use of the fact $m \in \mathbb{Z}$, so if we manage to prove the queried identity for some $m = q \in \mathbb{Q}$, then it will hold for all $m \in q + \mathbb{Z}$. For example, a Taylor expansion of $1/\sqrt{1-x}$ provides the induction base

$$
\frac{1}{(1-x)^{1/2}} = \sum_{k=0}^{\infty} \binom{k-1/2}{k} x^k,
$$

and hence the original formula holds for all $m \in \frac{1}{2}\mathbb{Z}$.

**Exercise 19** Prove the reflection formula for binomial coefficients

$$
\binom{\alpha}{k} = (-1)^k \binom{k-\alpha-1}{k}
$$

where $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$.

For $k < 0$ or $k > \alpha$ both sides evaluate to zero and there is nothing to prove. Otherwise, the right hand side does not vanish and we may divide the conjecture through it. Plugging in the definition of the binomial coefficient gives

$$
\begin{aligned}
\frac{\binom{\alpha}{k}}{\binom{k-\alpha-1}{k}} &= \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)/k!}{(k-\alpha-1)(k-\alpha-2)\ldots(k-\alpha-1-k+1)/k!} \\
&= \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{(-\alpha)\ldots(-(\alpha-k+2)(-(\alpha-k+1))} = (-1)^k
\end{aligned}
$$

as we wished to prove.

**Exercise 20** Consider the sequence $\{f_n\}_{n=0}^{\infty}$ defined by

$$
f_n = \sum_{k=0}^{n} \binom{m+k-1}{k} \binom{m}{n-k} (-1)^{n-k}
$$

where $m$ is an arbitrary but fixed complex number. Prove that $f_0 = 1$ and $f_n = 0$ for $n \neq 0$.

First note that for all $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$
\frac{\partial^n}{\partial^n x} \frac{1}{(1-x)^{\alpha}} = \alpha^{\overline{n}} \frac{1}{(1-x)^{\alpha+n}},
$$

in particular,

$$\frac{\partial^n}{\partial^n x} \frac{1}{(1-x)^\alpha}\Big|_{x=0} = \alpha^{\overline{n}},$$

and by Taylor's theorem

$$\frac{1}{(1-x)^\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^{\overline{n}}}{n!} x^n = \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} x^n. \tag{1}$$

Now consider the generating function $f(x)$ of $f_n$.

$$\begin{aligned}
f(x) = \sum_{n=0}^{\infty} f_n x^n &= \sum_{n=0}^{\infty} \Big( \sum_{k=0}^{n} \binom{m+k-1}{k} \binom{m}{n-k} (-1)^{n-k} \Big) x^n \\
&= \sum_{n=0}^{\infty} \Big( \sum_{k=0}^{n} \binom{m+k-1}{k} \binom{-m+(n-k)-1}{n-k} \Big) x^n \\
&= \sum_{n=0}^{\infty} \binom{m+n-1}{n} x^n \sum_{n=0}^{\infty} \binom{-m+n-1}{n} x^n \\
&\overset{(1)}{=} \frac{1}{(1-x)^m} \frac{1}{(1-x)^{-m}} = 1.
\end{aligned}$$

The claim follows by comparing coefficients.

*Remark:* Given the definition of $f$, Maple returns 0 ignoring the special case $n = 0$. Carsten's Sigma simplifies the sum to

$$-(-1)^n(-1+(-1)^{2n})\binom{m}{n}$$

which is clearly 0 for all $n \in \mathbb{Z}$. It also ignores the special case.

**Exercise 21** Study recurrences and generating functions of the Stirling numbers of the second kind, $S(n, k)$.

$S(n, k)$ is by definition the number of partitions of a set of $n$ elements into $k$ disjoint nonempty subset. For a recurrence of $S(n, k)$, consider a systematic way to enumerate all those partitions. Given $\{1, \dots, n\}$ we may first compute the partitions of $\{1, \dots, n-1\}$ into $k$ subsets and add $\{n\}$ to one of those. There are $S(n-1, k)$ partitions of $\{1, \dots, n-1\}$ and $k$ possible extensions to partitions of $\{1, \dots, n\}$. There are further partitions where $\{n\}$ is a singleton. These are not included yet because empty subsets are not permitted. There are $S(n-1, k-1)$ partitions $S_1 \cup \dots \cup S_{k-1}$ of $\{1, \dots, n-1\}$ and they extend to as many partitions $S_1 \cup \dots \cup S_{k-1} \cup \{n\}$ of $\{1, \dots, n\}$. We have found the recurrence

$$S(n, k) = kS(n-1, k) + S(n-1, k-1) \qquad (n \geq 1, k \geq 1)$$

The initial values are $S(n, 0) = 0$ $(n \neq 0)$, $S(0, k) = 0$ $(k \neq 0)$ and $S(0, 0) = 1$. We extend the definition of $S$ by putting $S(n, k) := 0$ $(n < 0$ or $k < 0)$. The recurrence then holds for all $(n, k) \in \mathbb{Z} \times \mathbb{Z}$ except for $(n, k) = (0, 0)$.

We can now study the generating functions of $S$.

First consider the bivariate generating function $F(x,y)$.

$$\begin{aligned}
F(x,y) &= \sum_{(n,k)} S(n,k)x^n y^k = 1+\sum_{(n,k)\neq(0,0)} kS(n-1,k)x^n y^k + \sum_{(n,k)\neq(0,0)} S(n-1,k-1)x^n y^k \\
&= 1+\sum_{(n,k)} kS(n-1,k)x^n y^k + \sum_{(n,k)} S(n-1,k-1)x^n y^k \\
&= 1+xy\frac{\partial}{\partial y}\sum_{(n,k)} S(n,k)x^n y^k + xy\sum_{(n,k)} x^n y^k \\
&= 1+xy\frac{\partial}{\partial y}F(x,y)+xyF(x,y).
\end{aligned}$$

As a solution for this system, Maple outputs a complicated expression involving the incomplete Gamma function which we do not want to reproduce here. However, there is a nice form of the "semi exponential" bivariate generating function $\tilde{F}(x,y)$ which is exponential wrt. $n$ and ordinary wrt. $k$. Making use of the results of Exercise 23 below we get

$$\begin{aligned}
\tilde{F}(x,y) &= \sum_{(n,k)} \frac{S(n,k)}{n!}x^n y^k = \sum_{k=0}^{\infty}\Big(\sum_{n=0}^{\infty}\frac{S(n,k)}{n!}x^n\Big)y^k \\
&= \sum_{k=0}^{\infty}\frac{(\mathrm{e}^x-1)^k}{k!}y^k = \exp(y(\mathrm{e}^x-1)).
\end{aligned}$$

Now keep $n$ discrete and consider the generating function $G_n(y)$. For $n>0$ we have

$$\begin{aligned}
G_n(y) &= \sum_{k=0}^{\infty} S(n,k)y^k = \sum_{k=0}^{\infty} kS(n-1,k)y^k + \sum_{k=0}^{\infty} S(n-1,k-1)y^k \\
&= y\frac{\partial}{\partial y}\sum_{k=0}^{\infty} S(n-1,k)y^k + y\sum_{k=1}^{\infty} S(n-1,k-1)y^{k-1} \\
&= y\frac{\partial}{\partial y}G_{n-1}(y)+yG_{n-1}(y)
\end{aligned}$$

Is there a closed form for $G_n$? We note that $G_n$ is holonomic, the defining annihilating operator is $S_n - yD_y - y$. Together with $G_0(y)=1$ the $G_n(y)$ can be consecutively computed for all $n>0$, so the system has a unique solution. We do, however, not know if this solution admits a closed form. (Exercise 23 below will consider the exponential sister of $G_n(x)$)

Now keep $k$ discrete and consider the generating function $H_k(x)$. For $k>0$ we have

$$\begin{aligned}
H_k(x) &= \sum_{n=0}^{\infty} S(n,k)x^n = \sum_{n=0}^{\infty} kS(n-1,k)x^n + \sum_{n=0}^{\infty} S(n-1,k-1)x^n \\
&= kx\sum_{n=0}^{\infty} S(n,k)x^n + x\sum_{n=0}^{\infty} S(n,k-1)x^n = kxH_k(x)+xH_{k-1}(x),
\end{aligned}$$

Therefore

$$\begin{aligned}
H_k(x) &= \frac{x}{1-kx}H_{k-1}(x) = \frac{x}{1-kx}\,\frac{x}{1-(k-1)x}H_{k-2}(x) = \cdots \\
&= H_0(x)\prod_{i=1}^{k}\frac{1}{\frac{1}{x}-i} = \frac{1}{(\frac{1}{x})^{\underline{k}}}\,.
\end{aligned}$$

Note that $H_0(x) = \sum_{n=0}^{\infty} S(n,0)x^n = 1$.
Is there a closed form for $S(n,k)$? We once more make use of the result of Exercise 23. We have

$$S(n,k) = n!\langle x^n\rangle \frac{(\mathrm{e}^x - 1)^k}{k!} = \frac{1}{k!}\frac{\partial^n}{\partial^n x}(\mathrm{e}^x - 1)^k\Big|_{x=0}$$
$$\frac{\partial^n}{\partial^n x}(\mathrm{e}^x - 1)^k = \frac{\partial^n}{\partial^n x}\sum_{i=0}^{k}\binom{k}{i}\mathrm{e}^{ix}(-1)^{k-i} = \sum_{i=0}^{k}\binom{k}{i} i^n \mathrm{e}^{ix}(-1)^{k-i}$$

and therefore

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i} i^n(-1)^{k-i}.$$

This may or may not be considered a closed form.

**Exercise 22** Find closed form expressions for the following sequences and their ordinary and exponential generating functions.

a) $a_{n+1} = 3a_n + 2$ $(n \geq 0)$, $a_0 = 0$

b) $a_{n+2} = 2a_{n+1} - a_n$ $(n \geq 0)$, $a_0 = 0$, $a_1 = 1$

c) $a_{n+1} = \frac{1}{3}a_n + 1$ $(n \geq 0)$, $a_0 = 0$

Find closed form representations for the exponential generating function of the following sequences.

d) $a_{n+1} = (n+1)a_n - \binom{n}{2}a_{n-2}$ $(n \geq 0)$, $a_0 = 1$

e) $2a_{n+1} = \sum_{k=0}^{n}\binom{n}{k}a_k a_{n-k}$ $(n \geq 1)$, $a_0 = a_1 = 1$

f) $2a_{n+1} = \sum_{k=0}^{n}\binom{n}{k}a_k a_{n-k}$ $(n \geq 0)$, $a_0 = 1$.

**Ad a)** The closed form for $a_n$ itself is easily obtained as $a_n = 3^n - 1$ as described in detail in previous exercises on similar examples.
This closed form admits immediately closed forms for the generating functions:

$$\begin{aligned}
a(x) &= \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(3^n - 1)x^n = \sum_{n=0}^{\infty}(3x)^n - \sum_{n=0}^{\infty} x^n \\
&= \frac{1}{1-3x} + \frac{1}{1-x} = \frac{2x}{(3x-1)(x-1)} \\
\bar{a}(x) &= \sum_{n=0}^{\infty}\frac{a_n}{n!}x^n = \sum_{n=0}^{\infty}\frac{3^n - 1}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}(3x)^n - \sum_{n=0}^{\infty}\frac{1}{n!}x^n = \mathrm{e}^{3x} - \mathrm{e}^x.
\end{aligned}$$

**Ad b)** Again the defining recurrence is C-finite, so we obtain $a_n = n$ effordlessly.
As for the generating functions, we have

$$a(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n x^n = \sum_{n=1}^{\infty} n x^n = x\sum_{n=1}^{\infty} n x^{n-1}$$
$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{\partial}{\partial x}\sum_{n=1}^{\infty} x^n = \frac{\partial}{\partial x}\Big(\frac{1}{x-1} - 1\Big) = \frac{1}{(x-1)^2}$$
$$\implies \quad a(x) = \frac{x}{(x-1)^2}$$

and

$$\begin{aligned}\bar{a}(x) &= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = x \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} \\ &= x \sum_{n=0}^{\infty} \frac{1}{n!} x^n = x\mathrm{e}^x.\end{aligned}$$

**Ad c)** The closed form for $a_n$ is again for free: $a_n = \frac{3}{2}(1-(1/3)^n)$. What are the generating functions?

$$\begin{aligned}a(x) &= \sum_{n=0}^{\infty} a_n x^n = \tfrac{3}{2} \sum_{n=0}^{\infty} (1-3^{-n}) x^n = \tfrac{3}{2} \Big( \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (x/3)^n \Big) \\ &= \tfrac{3}{2} \Big( \frac{1}{x-1} + \frac{3}{3-x} \Big) = \frac{3x}{(x-1)(x-3)}, \\ \bar{a}(x) &= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \tfrac{3}{2} \Big( \sum_{n=0}^{\infty} \frac{1}{n!} x^n - \sum_{n=0}^{\infty} \frac{1}{n!} (x/3)^n \Big) = \tfrac{3}{2} (\mathrm{e}^x - \mathrm{e}^{x/3})\end{aligned}$$

*Remark:* All generating functions in a), b), and c) can be obtained as well be the standard method without using the closed form of $a_n$, but the required "manipulatorics" is much more involved.

**Ad d)** Starting from the recurrence of $a_n$, we get

$$\begin{aligned}a_{n+1} &= (n+1) a_n - \tfrac{1}{2} n(n-1) a_{n-2} \\ \frac{a_{n+1}}{n!} &= (n+1) \frac{a_n}{n!} - \tfrac{1}{2} n(n-1) \frac{a_{n-2}}{n!} \\ \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n &= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n - \tfrac{1}{2} \sum_{n=0}^{\infty} n(n-1) \frac{a_{n-2}}{n!} x^n.\end{aligned}$$

We observe

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n &= \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{a_n}{n!} n x^{n-1} = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \bar{a}'(x) \\ \sum_{n=0}^{\infty} \frac{a_n}{n!} (n+1) x^n &= \frac{\partial}{\partial x} \sum_{n=0}^{\infty} \frac{a_n}{n!} x^{n+1} = \frac{\partial}{\partial x} (x \bar{a}(x)) = \bar{a}(x) + x \bar{a}'(x) \\ \sum_{n=0}^{\infty} n(n-1) \frac{a_{n-2}}{n!} x^n &= \sum_{n=2}^{\infty} n(n-1) \frac{a_{n-2}}{n!} x^n = x^2 \sum_{n=2}^{\infty} \frac{a_{n-2}}{(n-2)!} x^{n-2} \\ &= x^2 \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = x^2 \bar{a}(x).\end{aligned}$$

This yields a differential equation for $\bar{a}(x)$

$$\begin{aligned}\bar{a}'(x) &= \bar{a}(x) + x \bar{a}'(x) - \tfrac{1}{2} x^2 \bar{a}(x) \\ (1-x) \bar{a}'(x) &= (1 - \tfrac{1}{2} x^2) \bar{a}(x).\end{aligned}$$

According to Maple, a generic solution of this equation is

$$\bar{a}(x) = \alpha \frac{\exp(\frac{1}{4} x^2 + \frac{1}{2} x)}{\sqrt{x-1}}.$$

By $\bar{a}(0) = -\mathrm{i}\alpha \stackrel{!}{=} 1 = a_0$ we obtain $\alpha = \mathrm{i}$ and therefore

$$\bar{a}(x) = \frac{\exp(\frac{1}{4}x^2 + \frac{1}{2}x)}{\sqrt{1-x}}.$$

**Ad e)** We first note that

$$\sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} = n! \sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!}.$$

Let $\tilde{a}(x)$ be the exponential generating function of $a_n$. Then

$$\tilde{a}(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = 1 + x + \sum_{n=2}^{\infty} \frac{a_n}{n!} x^n. \tag{2}$$

Consider the last sum.

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{a_n}{n!} x^n &= \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{a_{n+1}}{(n+1)!} x^{n+1} = \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!} x^{n+1} \\
&= \tfrac{1}{2} \int \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!} x^n \, dx = \tfrac{1}{2} \int \Big( \Big( \sum_{n=0}^{\infty} a_n x^n \Big)^2 - a_0^2 \Big) \, dx \\
&= \tfrac{1}{2} \int (\tilde{a}(x) - 1) \, dx.
\end{aligned}$$

Plugging this representation into (2) and differentiating on both sides gives

$$\frac{\partial}{\partial x} \tilde{a}(x) = 1 + \tfrac{1}{2}(\tilde{a}(x)^2 - 1) = \tfrac{1}{2}\tilde{a}(x) + \tfrac{1}{2}.$$

The solution of this equation is $\tilde{a}(x) = \tan \frac{x+\alpha}{2}$ for some $\alpha$. By $\tilde{a}(0) = a_0 = 1$ we obtain $\alpha = \pi/2$ and finally

$$\tilde{a}(x) = \tan \frac{2x + \pi}{4}$$

for the required generating function.

**Ad f)** We first note that

$$\sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} = n! \sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!}.$$

Using this we obtain from the recurrence

$$\begin{aligned}
2a_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} a_k a_{n-k} \\
2\frac{a_{n+1}}{n!} &= \sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!} \\
2\sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} x^n &= \sum_{n=0}^{\infty} \Big( \sum_{k=0}^{n} \frac{a_k}{k!} \frac{a_{n-k}}{(n-k)!} \Big) x^n \\
2\bar{a}'(x) &= \Big( \sum_{n=0}^{\infty} a_n x^n \Big)^2 = \bar{a}(x)^2
\end{aligned}$$

It follows that $\bar{a}(x) = 2/(\alpha - x)$ for some $\alpha$. $\bar{a}(0) = 2/\alpha \stackrel{!}{=} 1 = a_0$ gives $\alpha = 2$ and hence the generating function is

$$\bar{a}(x) = \frac{2}{2-x}.$$

**Exercise 23** Compute a closed form representation for

$$f_k(x) := \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} x^n$$

For $k > 0$ we have

$$f_k(x) = \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} x^n = k \sum_{n=k}^{\infty} \frac{S(n-1,k)}{n!} x^n + \sum_{n=k}^{\infty} \frac{S(n-1,k-1)}{n!} x^n. \tag{3}$$

We observe the following relations.

$$\begin{aligned}
\sum_{n=k}^{\infty} \frac{S(n-1,k-1)}{n!} x^n &= \int \sum_{n=k}^{\infty} \frac{S(n-1,k-1)}{(n-1)!} x^{n-1}\, dx \\
&= \int \sum_{n=k-1}^{\infty} \frac{S(n,k-1)}{n!} x^n \, dx = \int f_{k-1}(x)\, dx \\
\sum_{n=k}^{\infty} \frac{S(n-1,k)}{n!} x^n &= \int \sum_{n=k}^{\infty} \frac{S(n-1,k)}{(n-1)!} x^{n-1}\, dx = \int \sum_{n=k-1}^{\infty} \frac{S(n,k)}{n!} x^n \, dx \\
&= \int \Big( \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} x^n + \underbrace{\frac{S(k-1,k)}{(k-1)!} x^{k-1}}_{=0} \Big)\, dx \\
&= \int f_k(x)\, dx
\end{aligned}$$

Plugging them into (3) and differentiating both sides gives

$$\begin{aligned}
f_k(x) &= k \int f_k(x)\, dx + \int f_{k-1}(x)\, dx \\
\frac{\partial}{\partial x} f_k(x) &= k f_k(x) + f_{k-1}(x).
\end{aligned}$$

We regard the latter as inhomogenous differential equation in $f_k$. The corresponding homogenous equation has the solution $h_k(x) = c_k \mathrm{e}^{kx}$. Variation of the constant, $f_k(x) := c_k(x)\mathrm{e}^{kx}$, leads to

$$\begin{aligned}
\frac{\partial}{\partial x} c_k(x)\mathrm{e}^{kx} - k c_k(x)\mathrm{e}^{kx} &= c_{k-1}\mathrm{e}^{(k-1)x} \\
c_k'(x)\mathrm{e}^{kx} + c_k(x) k \mathrm{e}^{kx} - k c_k(x)\mathrm{e}^{kx} &= c_{k-1}\mathrm{e}^{(k-1)x} \\
c_k'(x) &= c_{k-1}(x)\mathrm{e}^{-x} \\
c_k(x) &= \int_0^x c_{k-1}(t)\mathrm{e}^{-t}\, dt
\end{aligned} \tag{4}$$

where the last step takes into accout that $f_k(0) = c_k(0)\mathrm{e}^0 = 0$ for $k > 0$.

Now consider $k = 0, 1, 2, \ldots$. For $k = 0$ we have $f_0(x) = c_0(x)\mathrm{e}^{0x} = 1$, so $c_0(x) = 1$. The further $f_k(x)$ can be obtained by means of (4):

$$\begin{aligned}
c_1(x) &= \int_0^x c_0(t)\mathrm{e}^{-t}\,dt = 1 - \mathrm{e}^{-x} && \Longrightarrow \quad f_1(x) = \mathrm{e}^x - 1,\\
c_2(x) &= \int_0^x c_1(t)\mathrm{e}^{-t}\,dt = \tfrac{1}{2}(1 - \mathrm{e}^{-x})^2 && \Longrightarrow \quad f_2(x) = \tfrac{1}{2}(\mathrm{e}^x - 1)^2,\\
c_3(x) &= \int_0^x c_2(t)\mathrm{e}^{-t}\,dt = \tfrac{1}{6}(1 - \mathrm{e}^{-x})^3 && \Longrightarrow \quad f_3(x) = \tfrac{1}{6}(\mathrm{e}^x - 1)^3,\\
c_4(x) &= \int_0^x c_3(t)\mathrm{e}^{-t}\,dt = \tfrac{1}{24}(1 - \mathrm{e}^{-x})^4 && \Longrightarrow \quad f_4(x) = \tfrac{1}{24}(\mathrm{e}^x - 1)^4, \ldots
\end{aligned}$$

This leads to the conjecture

$$c_k(x) = \frac{(1 - \mathrm{e}^{-x})^k}{k!}, \qquad \text{i.e.,} \quad f_k(x) = \frac{(\mathrm{e}^x - 1)^k}{k!}$$

which we will now prove by induction. We have seen enough base cases above. Assume now $c_k(x)$ is of the desired form. Then

$$\begin{aligned}
c_{k+1}(x) &= \int_0^x c_k(t)\mathrm{e}^{-t}\,dt = \frac{1}{k!}\int_0^x (1 - \mathrm{e}^{-t})^k \mathrm{e}^{-t}\,dt = \frac{1}{k!}\int_{-1}^{-\mathrm{e}^{-x}} (1 + u)^k\,du\\
&= \frac{1}{k!}\left[\frac{1}{k+1}(1 + u)^{k+1}\right]_{u=-1}^{-\mathrm{e}^{-x}} = \frac{1}{(k+1)!}\Big((1 - \mathrm{e}^{-x})^{k+1} - 0\Big)\\
&= \frac{(1 - \mathrm{e}^{-x})^{k+1}}{(k+1)!}.
\end{aligned}$$

**Exercise 24** Compute a closed form representation for the exponential generating function of the bell numbers $B(n)$ defined by

$$B(n) = \sum_{k=0}^{n} S(n, k) \qquad (n \geq 0)$$

We will use that $S(n, k) = 0$ if $k > n$:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} S(n, k) x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{S(n, k)}{n!} x^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{S(n, k)}{n!} x^n\\
&\overset{(*)}{=} \sum_{k=0}^{\infty} \frac{(\mathrm{e}^x - 1)^k}{k!} = \exp(\mathrm{e}^x - 1),
\end{aligned}$$

where $(*)$ is justified by the previous exercise.

**Exercise 25** Let $c(n, k)$ be the number of permutations $\pi \in S_n$ with exactly $k$ cycles. Give a combinatorial argument for the recurrence

$$c(n, k) = (n - 1)c(n - 1, k) + c(n - 1, k - 1) \qquad (n, k \geq 1)$$

In order to enumerate all permutations $\pi$ of $\{1, \ldots, n\}$, we proceed recursively.

The first bunch of permutations leaves $n$ fixed. Then $(n)$ is one cycle of its own, and there are $k - 1$ further cycles left for permuting the remaining $n - 1$ elements. These gives $c(n - 1, k - 1)$ permutations.

Next we consider a permutation $\pi$ of $\{1,\ldots,n-1\}$ with $k$ cycles. There are $c(n-1,k)$ permutations of this type. Let

$$\pi = (p_1\ p_2 \ldots p_{c_1})(p_{c_1+1} \ldots p_{c_2}) \ldots\ldots (p_{c_{k-1}+1} \ldots p_{c_k})$$

be a representation of the permutation $\pi$. Note that $c_k = n-1$. Any extension of $\pi$ to a permutation of $\{1,\ldots,n\}$ also having exactly $k$ cycles must have the form

$$\pi = (p_1 \ldots) \ldots (\ldots p_i\ n\ p_{i+1} \ldots) \ldots (\ldots p_{n-1})$$

for some $i = 1,\ldots,n-1$. If $p_i$ and $p_{i+1}$ belong to different cycles, we may put $n$ always to the cycle of $p_i$. This is without loss of generality because $(a_1\ a_2 \ldots a_\ell) = (a_\ell\ a_1\ a_2 \ldots a_{\ell-1})$. We have found that there are exactly $n-1$ ways to extend $\pi$ to a permutation of $\{1,\ldots,n\}$ leaving the number of cycles fixed.

Putting things together we obtain $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$, as claimed.

**Exercise 26** Prove

$$x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}}, \quad x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k \qquad (n \in \mathbb{N}_0)$$

where $S(n,k)$ and $s(n,k)$ denote the Striling numbers of the second and first kind, respectively.

(1) For $n=0$ there is nothing to prove. Suppose the formula holds for some $n \geq 0$. Then

$$\begin{aligned}
&\sum_{k=0}^{n+1} S(n+1,k)x^{\underline{k}} = \sum_{k=0}^{n+1} kS(n,k)x^{\underline{k}} + \sum_{k=0}^{n+1} S(n,k-1)x^{\underline{k}} \\
&= \sum_{k=0}^{n} S(n,k)x^{\underline{k}} + \sum_{k=-1}^{n} S(n,k)x^{\underline{k+1}} = \sum_{k=0}^{n} S(n,k)x^{\underline{k}} + \sum_{k=0}^{n} S(n,k)x^{\underline{k}}(x-k) \\
&= \sum_{k=0}^{n} S(n,k)x^{\underline{k}} + x\sum_{k=0}^{n} S(n,k)x^{\underline{k}} - \sum_{k=0}^{n} S(n,k)x^{\underline{k}} = x^{n+1},
\end{aligned}$$

and we are done.

(2) We consider the generating function $G_n(y)$ of $s(n,k)$. The recurrence for $c(n,k)$ together with the definition of $s(n,k)$ gives the recurrence

$$s(n,k) = (1-n)s(n-1,k) + s(n-1,k-1) \qquad (n,k \geq 1)$$

Note that $s(n,k) = 0$ for $k > n$, so it suffices to show the claim for the generating function in place of the finite sum. For $n=0$ we have $G_0(y) = 1$. For $n > 0$ we have

$$\begin{aligned}
&G_n(y) = \sum_{k=0}^{\infty} s(n,k)y^k = (1-n)\sum_{k=0}^{\infty} s(n-1,k)x^k + \sum_{k=0}^{\infty} s(n-1,k-1)x^k \\
&= (1-n)G_{n-1}(y) + \sum_{k=1}^{\infty} s(n-1,k-1)x^k = (1-n)G_{n-1}(y) + x\sum_{k=0}^{\infty} s(n-1,k)x^k \\
&= (y+1-n)G_{n-1}(y).
\end{aligned}$$

This gives

$$\begin{aligned}
G_1(y) &= y = y^{\underline{1}},\\
G_2(y) &= y(y-1) = y^{\underline{2}},\\
G_3(y) &= y(y-1)(y-2) = y^{\underline{3}}, \dots\\
G_k(y) &= y(y-1)\cdots(y-(k-1)) = y^{\underline{k}}.
\end{aligned}$$

**Exercise 27** Is $\mathbb{C}[\![x]\!]$ a Eucledian domain?

Yes: We have to provide a degree function $\deg\colon \mathbb{C}[\![x]\!]\setminus\{0\} \to \mathbb{N}_0$ such that $\deg ab \geq \deg a$ for all $a, b \neq 0$ and such that for all $a, b \in \mathbb{C}[\![x]\!]$, $b \neq 0$ there exists elements $q, r \in \mathbb{C}[\![x]\!]$ with $a = qb + r$ where $r = 0$ or $\deg r < \deg b$.

For every nonzero power series $a = \sum_{n=0}^{\infty} a_n x^n$ put $\deg a := \min\{\, n \in \mathbb{N}_0 : a_n \neq 0 \,\}$. Let $a, b \neq 0$ be arbitrary power series. Then $a = x^{\deg a}\tilde{a}$, $b = x^{\deg b}\tilde{b}$ where $\tilde{a}, \tilde{b}$ are invertible. (We use the statement of Exercise 29.)

Apparently we have $\deg \tilde{a} = \deg \tilde{b} = 0 = \deg \tilde{a}\tilde{b}$, the latter "=" because $\mathbb{C}[\![x]\!]$ is an integral domain. It follows that $\deg ab = \deg a + \deg b \geq \deg a$, as required.

Now, assume $\deg a \geq \deg b$ and define $q = \frac{\tilde{a}}{\tilde{b}} x^{\deg a - \deg b}$ (recall that $\tilde{b}$ is invertible) and $r = 0$. Then

$$qb + r = qb = \frac{\tilde{a}}{\tilde{b}} x^{\deg a - \deg b} \cdot x^{\deg b}\tilde{b} + 0 = \tilde{a} x^{\deg a} = a,$$

as required. On the other hand, if $\deg a < \deg b$, define $q = 0$ and $r = a$. Then again $a = qb + r$ and $\deg r < \deg b$.

**Exercise 28** What is the quotient field of $\mathbb{C}[\![x]\!]$?

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[\![x]\!]$ be a power which does not have an inverse ($a(x) \neq 0$). Let $N$ be the least number $N \in \mathbb{N}$ such that $a_N \neq 0$. According to the next exercise, $N > 0$ and and $\tilde{a}(x) := \sum_{n=0}^{\infty} \tilde{a}_n x^n$ where $\tilde{a}_n := a_{n+N}$ is invertible.

We have $a(x) = x^N \tilde{a}(x)$. If there were an inverse $(x^N)^{-1}$ for $x^N$ then there were also an inverse for $a(x)$, namely $a^{-1}(x) = \tilde{a}^{-1}(x)(x^N)^{-1}$. This motivates the consideration of $\mathbb{C}(\!(x)\!) := \mathbb{C}[\![x]\!][Y]/\langle xY - 1\rangle$.

Indeed $\mathbb{C}(\!(x)\!)$ is a field which contains $\mathbb{C}[\![x]\!]$, and every element of $\mathbb{C}(\!(x)\!)$ can be written as the quotient of two power series. Hence $\mathbb{C}(\!(x)\!) = Q(\mathbb{C}[\![x]\!])$.

**Exercise 29** Prove: $\sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[\![x]\!]$ has a multiplicative inverse if and only if $a_0 \neq 0$.

"$\Longrightarrow$" Suppose $\sum_{n=0}^{\infty} a_n x^n$ has $\sum_{n=0}^{\infty} b_n x^n$ as its multiplicative inverse. Then

$$1 = \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \Big( \sum_{k=0}^{n} a_k b_{n-k} \Big) x^n.$$

Comparing coefficients gives $a_0 b_0 = 1$, and hence $a_0 \neq 0$.

"$\Longleftarrow$" Let $a = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series with $a_0 \neq 0$. We make an indetermined ansatz for the inverse $a^{-1} = \sum_{n=0}^{\infty} b_n x^n$. In order to be the inverse it must be satisfied

$$1 = \sum_{n=0}^{\infty} \Big( \sum_{k=0}^{n} a_k b_{n-k} \Big) x^n.$$

Comparing coefficients gives $a_0 b_0 = 1$, so $b_0 = 1/a_0$. Suppose the coefficients of $a^{-1}$ are known up to some $n \geq 0$. Then

$$\sum_{k=0}^{n+1} a_k b_{n+1-k} = 0 \qquad \Longrightarrow \qquad b_{n+1} = \frac{1}{a_0} \sum_{k=1}^{n+1} a_k b_{n+1-k}$$

shows that also the next coefficient exists. By induction, all coefficient exist and so $a^{-1}$ exists.