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Analytic Combinatorics

Homework 4

Exercise 10 a) Complete the generating function proof of Binet's identity which was outlined in the lecture

b) Derive Binet's identity by the general theorem on solving linear C-finite recurrences.

c) Prove $f_n \sim \frac{1}{\sqrt{5}} r_+^{n+1}$ in the notation of the lecture.

d) Prove $f_n = \lfloor \frac{1}{\sqrt{5}} r_+^{n+1} \rfloor$ in the notation of the lecture.

Ad a) We have seen in the lecture that

$$f(x) = \frac{1}{1 - x - x^2}$$

is a generating functions for the Fibonacci numbers f_n and that $1 - x - x^2 = (1 - r_+x)(1 - r_-x)$ where

$$r_+ = \tfrac{1}{2}(1 + \sqrt{5}), \qquad r_- = \tfrac{1}{2}(1 - \sqrt{5}).$$

We compute a partial fraction decomposition for $f(x)$.

$$\begin{aligned}
f(x) &= \frac{1}{1 - x - x^2} = \frac{\alpha}{1 - r_+x} + \frac{\beta}{1 - r_+x} \\
\alpha &= \left.\frac{1 - xr_-}{1 - x - x^2}\right|_{x=1/r_-} = \tfrac{1}{10}(\sqrt{5} + 1)\sqrt{5} \\
\beta &= \left.\frac{1 - xr_+}{1 - x - x^2}\right|_{x=1/r_+} = \tfrac{1}{10}(\sqrt{5} - 1)\sqrt{5}
\end{aligned}$$

This representation of f does now admit the application of the geometric series.

$$f(x) = \frac{\alpha}{1 - r_+x} + \frac{\beta}{1 - r_+x} = \alpha \sum_{n=0}^{\infty} r_+^n x^n + \beta \sum_{n=0}^{\infty} r_-^n x^n.$$

Comparing coefficients gives

$$f_n = \alpha r_+^n + \beta r_-^n = \frac{1}{\sqrt{5}} \frac{1 + \sqrt{5}}{2} r_+^n - \frac{1}{\sqrt{5}} \frac{1 - \sqrt{5}}{2} r_-^n = \frac{1}{\sqrt{5}} \big(r_+^{n+1} - r_-^{n+1}\big),$$

as desired.

Ad b) By definition, the Fibonacci sequence forms a solution of the C-finite recurrence

$$f_{n+2} - f_{n+1} - f_n = 0 \qquad (n \in \mathbb{N}).$$

According to the theorem of Exercise 9 (previous sheet) the solutions to this recurrence are given precisely by linear combinations of $\{r_+^n\}_n, \{r_-^n\}_n$ where r_+, r_- are the roots of the polynomial $X^2 - X - 1$, their values are as in part a).

We find the linear combination from the initial conditions of $\{f_n\}_n$.

$$\begin{aligned} f_0 &= 1 = \alpha r_+^0 + \beta r_-^0 = \alpha + \beta \\ f_1 &= 1 = \alpha r_+^1 + \beta r_-^1 = \alpha r_+ + \beta r_-. \end{aligned}$$

This linear system has the solution

$$\alpha = \tfrac{1}{2} + \tfrac{1}{10}\sqrt{5}, \qquad \beta = \tfrac{1}{2} - \tfrac{1}{10}\sqrt{5},$$

and it follows that

$$\begin{aligned} f_n &= \left(\tfrac{1}{2} + \tfrac{1}{10}\sqrt{5}\right) r_+^n + \left(\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}\right) r_-^n \\ &= \frac{1}{\sqrt{5}}\left(\tfrac{1}{2}\sqrt{5} + \tfrac{1}{2}\right) r_+^n + \frac{1}{\sqrt{5}}\left(\tfrac{1}{2}\sqrt{5} - \tfrac{1}{2}\right) r_-^n \\ &= \frac{1}{\sqrt{5}}\left(r_+^{n+1} - r_-^{n+1}\right), \end{aligned}$$

as expected.

Ad c) By $|r_-| = \frac{1}{2}|1-\sqrt{5}| = \frac{1}{2}(\sqrt{5}-1) < \frac{1}{2}\sqrt{5}$ we get $|r_-/\sqrt{5}| < 1$ and therefore

$$f_n = \frac{1}{\sqrt{5}}(r_+^{n+1} + r_-^{n+1}) = \frac{1}{\sqrt{5}} r_+^{n+1} + \mathrm{o}(1).$$

The claim follows immediately.

Ad d) This is not true:

$$\left\lfloor \frac{r_+^0}{\sqrt{5}} \right\rfloor = 0 \neq 1 = f_0.$$

However, from part c) we know

$$f_n = \frac{r_+^{n+1}}{\sqrt{5}} + \mathrm{o}(1), \qquad \text{i.e.} \qquad f_n - \frac{r_+^{n+1}}{\sqrt{5}} \xrightarrow{n\to\infty} 0.$$

It is also immediate that this convergence is monotonous. Noting further that $|f_0 - r_+^1/\sqrt{5}| < 0.3$ (which can easily be checked numerically), we conclude that f_n is the closest integer to $r_+^{n+1}/\sqrt{5}$.

This is in accordance with (6.124) of [GKP89].

Exercise 11 Look up Strassen's algorithm.

Done. But I will not reproduce it here in order to save time and paper.

Exercise 12 Prove

$$\binom{a}{k} = \binom{a-1}{k} + \binom{a-1}{k-1}$$

for $a \in \mathbb{C}$ and $k \in \mathbb{Z}$.

Let $a \in \mathbb{C}$ and $k \in \mathbb{Z}$. We start a straightforward computation:

$$\begin{aligned} \binom{a-1}{k} &= \frac{(a-1)(a-2)\cdots(a-k+1)(a-k)}{k!} \\ \binom{a-1}{k-1} &= \frac{(a-1)(a-2)\ldots(a-k+1)\ k}{(k-1)!\ k} \\ \binom{a-1}{k} + \binom{a-1}{k-1} &= \frac{(a-1)(a-2)\ldots(a-k+1)\big((a-k)+k\big)}{k!} = \binom{a}{k}. \end{aligned}$$

If $k \notin \mathbb{Z}$, then $k-1 \notin \mathbb{Z}$ and so

$$\binom{a}{k} = \binom{a-1}{k} = \binom{a-1}{k-1} = 0,$$

so the identity also holds for $k \notin \mathbb{Z}$.

Exercise 13 Prove that for each given polynomial $f(k)$ there exists a polynomial $g(k)$ such that $f(k) = g(k+1) - g(k)$ for all integers k.

Let f_1, f_2 be arbitrary polynomials and α a constant. If g_1, g_2 are polynomials with the desired property for f_1, f_2, respectively, then $g_1 + g_2$ is such a polynomial for $f_1 + f_2$ and αg_1 is such a polynomial for αf_1. Therefore, it suffices to show the claim for $f(k) = k^n$ $(n \in \mathbb{N}_0)$.
Ansatz. Let $g(k) = \sum_{i=0}^{n+1} \gamma_i k^i$ and show that there are coefficients γ_i such that

$$g(k+1) - g(k) = \sum_{i=0}^{n+1} \gamma_i((k+1)^i - k^i) = k^n.$$

As the leading terms on the left hand side are easily seen to cancel out, we are left with a linear system of $n+1$ equations in $n+2$ unknowns. The homogenous part of the system must have a nontrivial solution. Observe that the homogenous system has the form

$$A \cdot x = (1, 0, 0, \ldots, 0)^T$$

and this is only unsolvable if $A = 0$ which is not the case. This completes the proof.