Manuel Kauers
RISC-Linz

Hagenberg, October 14, 2003

Analytic Combinatorics

Homework 2

Exercise 3 a) Find a solution of the differential equation

$$(1-z)^3 f'(z) - 2z - 2(1-z)^2 f(z) = 0, \quad f(0) = 0$$

by using your favorite Computer Algebra System.

b) Show that the solution of Mathematica,

$$f[z] \to \frac{-2(-I \cdot Pi + z + \mathrm{Log}[-1+z])}{(-1+z)^2}$$

"simplifies to"

$$g(z) = -\frac{2z}{(1-z)^2} - \frac{2}{(1-z)^2}\log(1-z)$$

Ad a)

```
(1-z)^3*diff(f(z),z) - 2*z - 2*(1-z)^2*f(z) = 0;
```

$$(1-z)^3\left(\frac{\partial}{\partial z} f(z)\right) - 2z - 2(1-z)^2 f(z) = 0$$

```
dsolve([%, f(0)=0], f(z));
```

$$f(z) = \frac{-2z - 2\ln(-1+z) + 2I\pi}{(-1+z)^2}$$

```
simplify(%, ln, assume=real);
```

$$f(z) = \frac{-2z - 2\ln(-1+z) + 2I\pi}{(-1+z)^2}$$

Ad b) First we change to readable notation and clean up a bit.

$$\begin{aligned} f(z) - g(z) &= \left(\frac{2\mathrm{i}\pi}{(1-z)^2} - \frac{2z}{(1-z)^2} - \frac{2}{(1-z)^2}\log(-1+z)\right) \\ &\quad - \left(-\frac{2z}{(1-z)^2} - \frac{2}{(1-z)^2}\log(1-z)\right) \\ &= \frac{2\mathrm{i}\pi}{(1-z)^2} + \frac{2}{(1-z)^2}\log(1-z) - \frac{2}{(1-z)^2}\log(-1+z) \end{aligned}$$

We are done if we show that the last expression is identically zero. This follows, if we manage to prove the more general result

$$\log(x) - \log(-x) = \mathrm{i}\pi \qquad (x \in \mathbb{C} \setminus \{0\}).$$

We write $x = |x|\mathrm{e}^{\mathrm{i}\phi}$ and use Euler's formula $\mathrm{e}^{\pi\mathrm{i}} = -1$. It follows

$$\begin{aligned} \log(x) - \log(-x) = \log(x) - \log(-1 \cdot x) &= \log(|x|\mathrm{e}^{\mathrm{i}\phi}) - \log(\mathrm{e}^{\pi\mathrm{i}}|x|\mathrm{e}^{\mathrm{i}\phi}) \\ &= \log|x| + \mathrm{i}\phi - \pi\mathrm{i} - \log|x| - \mathrm{i}\phi = -\pi\mathrm{i}. \end{aligned}$$

As exp is periodic in the imaginary direction with period 2π, we can freely replace $\log x$ by $2k\pi\mathrm{i} + \log x$ for $k \in \mathbb{Z}$. This means we change the *branch* of log. Adding $2\pi\mathrm{i}$ to the last equation gives the desired result.