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Analytic Combinatorics

Homework 13

Exercise 50 Use Sister Celine's Algorithm to evaluate the sum $f(n) = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Write $F(n,k) = \binom{n}{k} x^k$.

$$\begin{aligned} a_{00}F(n,k) + a_{10}F(n+1,k) + a_{01}F(n,k+1) + a_{11}F(n+1,k+1) = 0 \\ a_{00} + a_{10}\frac{n+1}{n+1-k} + a_{01}\frac{n-k}{k+1} + a_{11}\frac{n+1}{k+1} = 0 \\ (a_{01}x - a_{00})k^2 + \\ + (a_{10}n + a_{10} - a_{01}x + a_{00}n - 2a_{01}xn - a_{11}xn - a_{11}x)k + \\ + (a_{11}xn^2 + a_{00}n + a_{00} + a_{11}x + a_{01}xn^2 + a_{01}xn + a_{10} + 2a_{11}xn) = 0 \end{aligned}$$

We arrive at the following linear system for the a_{ij}.

$$\begin{pmatrix} -1 & x & 0 & 0 \\ n & -x(1+2n) & n+1 & -x(n+1) \\ n+1 & xn(n+1) & n+1 & x(n+1)^2 \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{01} \\ a_{01} \\ a_{11} \end{pmatrix} = 0$$

The solution space is given by

$$(a_{00}, a_{01}, a_{10}, a_{11}) = (-tx, 0, -t, t)$$

with t as parameter. Chosing $t = -1$ we obtain the recurrence

$$\begin{aligned} xF(n,k) + F(n,k+1) - F(n+1,k+1) = 0 \\ xf(n) + f(n) - f(n+1) = 0 \\ f(n+1) = (x+1)f(n). \end{aligned}$$

Together with $f(0) = 1$ we obtain the desired result $f(n) = (1+x)^n$.

Exercise 51 Use Zeilberger's Algorithm to evaluate the sum $\sum_{k=0}^{n} k\binom{n}{k}$.

Write $F(n,k) = k\binom{n}{k}$. We are optimistic and try first order 0.
$f_k := F(n,k) = k\binom{n}{k}$. We have $\frac{f_{k+1}}{f_k} = \frac{n-k}{k}$, and therefore

$$f_{\mathrm{rat}}(x) = \frac{n-x}{x} = \frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)}$$

where we may take $p(x) = 1$, $q(x) = n - x$, $r(x) = x - 1$. It is easily seen that the Gosper equation

$$1 = (n-x)y(x+1) - (x-1)y(x)$$

does not have a polynomial solution $y(x)$ (compare coefficients).

We have been too optimistic. Next, we try order 1.

$f_k := a_0 F(n,k) + a_1 F(n+1,k)$. Now we have

$$f_{\mathrm{rat}}(x) = \frac{a_0(n-x)+a_1(n+1)}{a_0(n+1-x)+a_1(n+1)} \frac{n+1-x}{x} = \frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)}$$

and we may take $p(x) = a_0(n+1-x)+a_1(n+1)$, $q(x) = n+1-x$, $r(x) = x-1$. The Gosper equation thus reads

$$a_0(n+1-x)+a_1(n+1) = (n+1-x)y(x+1) - (x-1)y(x)$$

Hoping for a solution $y(x) = y_0$ of degree 0 and rearranging terms we are lead to

$$-a_0 x + (a_0+a_1)(n+1) = -2xy_0 + (n+2)y_0$$

which has $y_0 = 1$, $a_0 = 2$, $a_1 = -\frac{n}{n+1}$, for example, as an obvious solution. Proceeding in the course of Gosper's algorithm, we have

$$\begin{aligned} g_{\mathrm{rat}}(x) &= \frac{y(x+1)}{y(x)} \frac{q(x)}{r(x)} = \frac{n+1-x}{x-1} \\ g_k &= \frac{1}{g_{\mathrm{rat}}(k)-1} f_k = \frac{k-1}{n+1-k} F(n,k). \end{aligned}$$

This gives the certificate $R(n,k) = \frac{k-1}{n+1-k}$ and the recurrence

$$\begin{aligned} 2F(n,k) - \frac{n}{n+1} F(n+1,k) &= g_{k+1} - g_k \\ 2f(n) &= \frac{n}{n+1} f(n+1) \\ f(n+1) &= 2\frac{n+1}{n} f(n). \end{aligned}$$

Together with $f(1) = 1$ this confirms the result $f(n) = n2^{n-1}$ from the lecture.

Q. E. D. — Quite Enough Done.