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Analytic Combinatorics

Homework 12

Exercise 48 Find the bug in the description of the algorithm for computing a Gosper form of a rational function $f(x) = s(x)/t(x)$.

For your convenience, I provide the following verbatim quote of what I actually wrote to the blackboard (according to Laura's notes, including the bug).

Suppose given $f(x) = \frac{s(x)}{t(x)}$.
Initialize $p(x) = 1, q(x) = s(x), r(x+1) = t(x)$.

1 Find all critical $j \in \mathbb{N}$

$$\mathrm{Res}_x(q(x), r(x+j)) = \mathrm{poly}(j)$$

Critical j are positive integer roots of this polynomial

2 If there is no root, done!
Otherwise, take the smallest j. Then

$$\begin{aligned} d(x) &:= \gcd(q(x), r(x+j)) \\ q(x) &= \bar{q}(x)d(x) \\ r(x+j) &= \bar{r}(x+j)d(x) \\ \Longrightarrow \quad \frac{q(x)}{r(x+1)} &= \frac{\bar{q}(x)}{\bar{r}(x+1)} \frac{d(x)}{d(x-j+1)} \end{aligned}$$

3 Update:

$$\begin{aligned} p(x) &:= p(x)d(x-j+1) \\ q(x) &:= \bar{q}(x) \\ r(x+1) &:= \bar{r}(x+1) \end{aligned}$$

4 Goto 1.

This turned out not to give the expected result on the example, because of an index bug in the updating step: $d(x-j+1)$ should read $d(x-j)$. Sorry about that...

Exercise 49 Show that $\sum_{k=0}^{n} \binom{a}{k} k$ is not Gosper summable.

Apply Gosper's algorithm. Let $f_k = \binom{a}{k} k$, so

$$\frac{f_{k+1}}{f_k} = \frac{k+1}{k} \frac{\binom{a}{k+1}}{\binom{a}{k}} = \frac{k+1}{k} \frac{a(a-1)\dots(a-k)k!}{(k+1)!n(n-1)\dots(a-k+1)} = \frac{a-k}{k} =: f_{\mathrm{rat}}(k).$$

Regarding a as indeterminate, we obtain the Gosper form

$$f_{\mathrm{rat}}(x) = \frac{a-x}{x} = \frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)}$$

with $p(x) = 1, q(x) = n - x, r(x) = x - 1$. Suppose $y(x) \in \mathbb{C}(n)[x]$ is a solution of the Gosper equation

$$1 = (a-x)y(x+1) - (x-1)y(x).$$

Assuming $\deg y = d \geq 0$ we obtain

$$1 = -2\mathrm{lc}(y)x^{d+1} + \text{smaller terms}$$

where $\mathrm{lc}(y)$ denotes the leading coefficient of y. Comparing coefficients gives a contradiction, and hence such a y can not exist.

The claim follows.