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Analytic Combinatorics
Homework 11

Exercise 44 Prove: If the Gosper equation does not have a polynomial solution, then the telescoping equation does not have a hypergeometric solution.

See GKP, page 255f; or $A = B$, page 75ff.

Exercise 45 In the notation of the lecture, prove that $g_{k+1}/g_k = g_{\mathrm{rat}}(k)$ in Gosper's algorithm.

Let $f_k, p, q, r, g_{\mathrm{rat}}$ be as in the lecture. Then

$$\begin{aligned}
\frac{g_{k+1}}{g_k} &= \frac{f_{k+1}}{f_k} \cdot \frac{g_{\mathrm{rat}}(k) - 1}{g_{\mathrm{rat}}(k+1) - 1} \\
&= \frac{p(k+1)}{p(k)} \cdot \frac{q(k)}{r(k+1)} \cdot \frac{\frac{y(k+1)}{y(k)} \cdot \frac{q(k)}{r(k)} - 1}{\frac{y(k+2)}{y(k+1)} \cdot \frac{q(k+1)}{r(k+1)} - 1} \\
&= \frac{p(k+1)}{p(k)} \cdot \frac{q(k)}{r(k+1)} \cdot \frac{\frac{y(k+1)q(k) - y(k)r(k)}{y(k)r(k)}}{\frac{y(k+2)q(k+1) - y(k+1)r(k+1)}{y(k+1)r(k+1)}} \\
&= \frac{p(k+1)}{p(k)} \cdot \frac{q(k)}{r(k+1)} \cdot \frac{p(k)y(k+1)r(k+1)}{p(k+1)y(k)r(k)} \\
&= \frac{y(k+1)q(k)}{y(k)r(k)} = g_{\mathrm{rat}}(k).
\end{aligned}$$

Exercise 46 Use Gosper's Algorithm to evaluate the following sums:

a) $\displaystyle\sum_{k=0}^{n} q^k \quad (q \in \mathbb{C} \text{ fixed})$

b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{a}{k} \quad (a \in \mathbb{C} \text{ fixed})$

c) $\displaystyle\sum_{k=1}^{n} (-1)^k \frac{k}{4k^2 - 1}$

d) $\displaystyle\sum_{k=1}^{n} \frac{1}{4k^2 - 1}$

Ad a) Let $f_k = q^k$. Then $\frac{f_{k+1}}{f_k} = q$ and we may take

$$f_{\mathrm{rat}}(x) = q = \frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)}$$

with $p(x) = 1$, $q(x) = q$, $r(x) = 1$. The Gosper equation

$$1 = q\, y(x+1) - r(x)y(x)$$

has the solution $y(x) = \frac{1}{q-1}$. Hence we have

$$g_{\mathrm{rat}}(x) = \frac{\frac{1}{q-1} q}{\frac{1}{q-1} 1} = q$$

and therefore

$$g_k = \frac{1}{q-1} q^k.$$

It follows that

$$\sum_{k=0}^{n} q^k = g_{n+1} - g_0 = \frac{q^{n+1}-1}{q-1},$$

as it should.

Ad b) Let $f_k = (-1)^k \binom{a}{k}$. Then $\frac{f_{k+1}}{f_k} = \frac{k-a}{k+1}$ and we may take

$$f_{\mathrm{rat}}(x) = \frac{x-a}{x+1} = \frac{p(x+1)}{p(x)} \frac{q(x)}{r(x+1)}$$

with $p(x) = 1$, $q(x) = x - a$ and $r(x) = x$, assuming that a is not a negative integer. The Gosper equation

$$1 = (x-a) y(x+1) - x\, y(x)$$

has the solution $y(x) = -1/a$. Hence we have

$$g_{\mathrm{rat}}(x) = \frac{-\frac{1}{a}(x-a)}{-\frac{1}{a} x} = \frac{x-a}{x}$$

and therefore

$$g_k = \frac{1}{\frac{k-a}{k} - 1} (-1)^k \binom{a}{k} = \frac{k}{a} (-1)^{k+1} \binom{a}{k}.$$

It follows that

$$\sum_{k=0}^{n} (-1)^k \binom{a}{k} = g_{n+1} - g_0 = \frac{n+1}{a} (-1)^n \binom{a}{n+1} = (-1)^n \binom{a-1}{n}.$$

Ad c) Let $f_k = (-1)^k \frac{k}{4k^2-1}$. Then $\frac{f_{k+1}}{f_k} = -\frac{k+1}{k} \frac{2k-1}{2k+3}$ and we may take

$$f_{\mathrm{rat}}(x) = \frac{x+1}{x} \frac{1-2x}{2x+3}$$

with $p(x) = x$, $q(x) = 1 - 2x$, $r(x) = 2x + 1$. The Gosper equation

$$x = (1-2x) y(x+1) - (2x+1) y(x)$$

has the solution $y(x) = -1/4$. Hence we have

$$g_{\mathrm{rat}}(x) = \frac{-\frac{1}{4}(1-2x)}{-\frac{1}{4}(2x+1)} = \frac{1-2x}{2x+1}$$

and therefore

$$g_k = \frac{1}{\frac{1-2k}{2k+1} - 1} (-1)^k \frac{k}{4k^2-1} = \frac{1}{4} (-1)^{k-1} \frac{1}{2k-1}.$$

It follows that

$$\sum_{k=1}^{n}(-1)^k\frac{k}{4k^2-1} = g_{n+1}-g_1 = \tfrac{1}{4}\Big((-1)^n\frac{1}{2n+1}-1\Big).$$

Ad d) Let $f_k = \frac{1}{4k^2-1}$. then $\frac{f_{k+1}}{f_k} = \frac{k+1}{k}\frac{2k-1}{2k+3}$ and we may take

$$f_{\mathrm{rat}}(x) = \frac{x+1}{x}\frac{2x-1}{2x+3}$$

with $p(x)=x$, $q(x)=2x-1$, $r(x)=2x+1$. The Gosper equation

$$x = (2x-1)y(x+1)-(2x+1)y(x)$$

now has the solution $y(x) = -\frac{1}{2}x+\beta$ for arbitrary β. We are lazy and choosse $\beta=0$. Then we have

$$g_{\mathrm{rat}}(x) = \frac{-\frac{1}{2}(x+1)}{-\frac{1}{2}x}\frac{2x-1}{2x+1} = \frac{(x+1)(2x-1)}{x(2x+1)}$$

and therefore

$$g_k = \frac{1}{\frac{(k+1)(2k-1)}{k(2k+1)}-1}\frac{1}{4k^2-1} = \frac{k}{1-2k}$$

It follows that

$$\sum_{k=1}^{n}(-1)^k\frac{1}{4k^2-1} = g_{n+1}-g_1 = 1-\frac{n+1}{2n+1}.$$

Cool.

Exercise 47 Use Gosper's Algorithm to prove that the sequence H_n of harmonic numbers ($H_n = \sum\limits_{k=1}^{n} 1/k$) is not hypergeometric (and hence, in particular, not rational)

We try to evaluate $\sum\limits_{k=1}^{n}\frac{1}{k}$. Writing $f_k = 1/k$ we have $\frac{f_{k+1}}{f_k} = \frac{k}{k+1}$, so we may put put

$$f_{\mathrm{rat}}(x) = \frac{x}{x+1} = \frac{p(x+1)}{p(x)}q(x)r(x+1)$$

with $p(x)=1$, $q(x)=x$, $r(x)=x$. The Gosper equation

$$1 = xy(x+1)-xy(x)$$

can not have a polynomial solution, because pluging such a solution into the right hand side would necessarily yield a polynomial with constant term $0\neq 1$ on the right hand side. That's it.