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Analytic Combinatorics

Homework 10

Exercise 41 Using Mallinger's package, compute a recurrence for the sequence $(s(n))_{n=0}^{\infty}$ where

$$s(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{(-1)^k}{2^k}$$

along the guideline given in the lecture.

Consider $y(x) = \sum_{n=0}^{\infty} \binom{2k}{k}/2^k x^k$ and observe that

$$\langle x^n \rangle \frac{1}{1-x} y(\frac{x}{x-1}) = \underbrace{\sum_{k=0}^{n} \binom{n}{k} (-1)^k \underbrace{\langle x^k \rangle y(x)}_{=:f(k)}}_{=s(n)}$$

by Euler's transform. We use Mallinger's package for converting a recurrence for $f(k)$ into a recurrence for $s(n)$. Because of the close analogy to the presentation in the lecture, we will omit explanatory comments.

In[1]:= $<<$ GeneratingFunctions.m
GeneratingFunctions Package by Christian Mallinger — ©RISC Linz — V 0.68 (07/17/03)

In[2]:= $F[k\_] = \text{Binomial}[2 * k, k];$

In[3]:= $f[k+1]/f[k] - F[k+1]/F[k]$ // FullSimplify // Together // Numerator // Simplify

Out[3]= $$-(1+2k)f[k] + (1+k)f[1+k]$$

In[4]:= rec = % == 0;

In[5]:= RE2DE[{rec, $f[0] == 1, f[1] == 1$}, $f[k], y[x]$]

Out[5]= $$\{-y[x] - (-1+2x)y'[x] == 0, y[0] == 1\}$$

In[6]:= ACompose[%, $y[x] == x/(x-1), y[x]$]
CanAb::denom : Warning. The input equation will be multiplied by its denominator

Out[6]= $$\{y[x] - (-1+x^2)y'[x] == 0, y[0] == 1\}$$

In[7]:= DECauchy[%, $y[x] == 1/(1-x), y[x]$]
CanAb::denom : Warning. The input equation will be multiplied by its denominator

Out[7]= $$\{x\,y[x] + (-1+x^2)y'[x] == 0, y[0] == 1\}$$

In[8]:= DE2RE[%, $y[x], s[n]$]

Out[8]= $$\{(1+n)s[n] + (-2-n)s[2+n] == 0, s[0] == 1, s[1] == 0\}$$

This is the expected output.

Exercise 42 Let $y(x) \in \mathbb{C}[\![x]\!] \setminus \{0\}$. Show: If $y'(x)/y(x)$ is algebraic, then $1/y(x)$ is D-finite.

See Stanley, Ex. 6.60, p. 247 (Solution on p. 281). [Sorry for being lazy.]

Exercise 43 Show that

a) $y_1(x) = 1/\cos x$ and

b) $y_2(x) = \sqrt{\log(1+x^2)}$

are not D-finite.

Ad a) Put $y(x) = \cos x$. Then $y'(x)/y(x) = -\tan x$ is not algebraic, and hence $1/y(x) = 1/\cos x$ is not D-finite.

Ad b) Suppose otherwise. Then, writing

$$y(x) = \frac{1}{\sqrt{\log(1+x^2)}},$$

we get by the theorem that

$$\frac{y'(x)}{y(x)} = -\frac{x}{1+x^2}\frac{1}{\log(1+x^2)}$$

is algebraic. Then in particular $1/\log(1+x^2)$ is algebraic, and so there exist polynomials $Q_0(x), \dots, Q_m(x)$, $Q_m(x) \neq 0$ with

$$\sum_{k=0}^{m} \frac{Q_k(x)}{\log(1+x^2)^k} = 0.$$

Clearing denominators gives

$$\sum_{k=0}^{m} Q_k(x) \log(1+x^2)^{m-k} = 0. \tag{1}$$

Without loss of generality, we may assume $Q_m(0) \neq 0$, for otherwise, we can replace (1) by

$$\sum_{k=0}^{m} Q_k'(x) \log(1+x^2)^{m-k} + Q_k(x) 2x(m-k) \log(1+x^2)^{m-k-1}$$

until $Q_m(0) \neq 0$ which must ultimately happen because $Q_m \neq 0$.
Now,

$$0 = \sum_{k=0}^{m} Q_k(x) \log(1+x^2)^{m-k} = Q_m(x) + \underbrace{\log(1+x^2)}_{\to 0} \underbrace{\sum_{k=0}^{m-1} Q_k(x) \log(1+x^2)^{m-k-1}}_{\text{continous}}$$

$$\xrightarrow{x \to 0} Q_m(0)$$

leads to the desired contradiction.