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Analytic Combinatorics

Homework 1

Exercise 1 Adapt the way how Gauß obtained $\sum_{k=1}^{n} k = n(n+1)/2$ to find closed form solutions for the following sums

$$\text{a)}\quad \sum_{k=0}^{n}(2k+1) \qquad \text{b)}\quad \sum_{k=1}^{n} k^2 \qquad \text{c)}\quad \sum_{k=1}^{n} k^3$$

Ad a) We write add the summands twice, in inverse direction and group corresponding terms.

$$\begin{array}{ccccccccccccc} 1 & + & 3 & + & 5 & + \dots + & 2k{+}1 & + \dots + & 2n{+}1 \\ 2n{+}1 & + & 2n{-}1 & + & 2n{-}3 & + \dots + & 2n{+}1{-}2k & + \dots + & 1 \\ \hline 2n{+}2 & + & 2n{+}2 & + & 2n{+}2 & + \dots + & 2n{+}2 & + \dots + & 2n{+}2 \end{array}$$

The last line sums up to $(n+1)(2n+2) = 2(n+1)^2$, and we conclude

$$\sum_{k=0}^{n}(2k+1) = \tfrac{1}{2}2(n+1)^2 = (n+1)^2.$$

Ad b) If Gauß' method is to determine $\sum_{k=1}^{n} a_k$ through a linear combination of a_k and a_{n-k}, then I did not find a way to apply this method to $a_k = k^2$.

It is, however, no problem at all to find a closed form of $\sum_k k^2$ by elementary manipulations. Graham, Knuth, Patashnik provide seven ways to do it in Section 2.5, and here is a way which is *not* mentioned there.

$$\begin{aligned} (k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\ (k+1)^3 - k^3 &= 3k^2 + 3k + 1 \\ \sum_{k=1}^{n}\bigl((k+1)^3 - k^3\bigr) &= \sum_{k=1}^{n}\bigl(3k^2 + 3k + 1\bigr) \\ (n+1)^3 - 1 &= 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n \\ \sum_{k=1}^{n} k^2 &= \tfrac{1}{3}\Bigl((n+1)^3 - 1 - n - \tfrac{3}{2}n(n+1)\Bigr) \\ &= \tfrac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Ad c) Once closed forms for $\sum_k k^r$ are known for $r < r_0$ and r_0 is odd, then Gauß' method applies again in a straightforward way. (We present it for $r_0 = 3$.)

$$\begin{array}{ccccccccc} 1 & + & 2^3 & + \dots + & (k+1)^3 & + \dots + & n^3 \\ n^3 & + & (n-1)^3 & + \dots + & (n-k)^3 & + \dots + & 1 \\ \hline p(n,0) & + & p(n,1) & + \dots + & p(n,k) & + \dots + & p(n,n-1) \end{array}$$

where

$$p(n,k) = (k+1)^3 + (n-k)^3 = 3(n+1)k^2 + 3(1-n^2)k + (n^3+1).$$

It follows that

$$\begin{aligned}
2\sum_{k=1}^{n} k^3 &= \sum_{k=0}^{n-1} p(n,k) \\
&= 3(n+1)\sum_{k=0}^{n-1} k^2 + 3(1-n^2)\sum_{k=0}^{n-1} k + (n^3+1) \\
&= \tfrac{1}{2}(n+1)n(2n-1)(n-1) + \tfrac{3}{2}(1-n^2)n(n-1) + n(n^3+1) \\
&= \tfrac{1}{2}n^2(n+1)^2, \quad \text{i.e.} \\
\sum_{k=1}^{n} k^3 &= \tfrac{1}{4}n^2(n+1)^2.
\end{aligned}$$

Note that only the closed forms of $\sum_k k^r$ with $r < 3$ have been used. Note further that the method happens to fail for even exponent because

$$(n-k)^{\text{even}} = k^{\text{even}} + \text{lower terms},$$

whereas in

$$(n-k)^{\text{odd}} = -k^{\text{odd}} + \text{lower terms}$$

the $-k^{\text{odd}}$ cancels with the k^{odd} from the first row.

Exercise 2 Let $F(n) = 2(n+1)\mathrm{H}_n - 4n$. Given $\mathrm{H}_n \sim \log n$, prove that $F(n) \sim 2n\log n$.

By definition of $\sim$, we have $\lim\limits_{n\to\infty} \frac{\mathrm{H}_n}{\log n} = 1$. We have to show $\lim\limits_{n\to\infty} \frac{F(n)}{2n\log n} = 1$. Plugging in the definition of F, and using elementary calculus, we get

$$\lim_{n\to\infty} \frac{F(n)}{2n\log n} = \underbrace{\lim_{n\to\infty} \frac{n+1}{n}}_{=1} \underbrace{\lim_{n\to\infty} \frac{\mathrm{H}_n}{\log n}}_{=1} - 2 \underbrace{\lim_{n\to\infty} \frac{1}{\log n}}_{=0} = 1.$$

The claim follows. (I am not quite sure if this was really meant to be the exercise. Sorry if not.)