Some Sample Sequences in $\mathcal{C}$

- rational functions in n
- $n \mapsto \alpha^n$ ($\alpha \in \mathbb{Q}$)
- $n \mapsto \alpha^{\beta^n}$ ($\alpha \in \mathbb{Q}, \beta \in \mathbb{N}$)
- $n \mapsto \mathrm{F}(n)$
- $n \mapsto \Gamma(\alpha n + \beta)$ ($\alpha \in \mathbb{Q}$)
- $n \mapsto \sum_{k=1}^{n} \mathrm{H}(n)\binom{n}{k}$
- next week: Extend $\mathcal{C}$ to $\mathcal{C}^*$ including

$$n \mapsto \sqrt[5]{\mathrm{H}(n)^2 + 7\sqrt{n}}, \qquad n \mapsto \sqrt{n + \sqrt{(n-1) + \sqrt{(n-2) + \cdots + \sqrt{2 + \sqrt{1}}}}}$$

Some Closure Properties

- Let $f(n), g(n), h(n) \in \mathcal{C}$, $h(n) \neq 0$ for all n.

- $f + g, f \cdot g, f - g, \alpha f, f/h \in \mathcal{C}$
- $n \mapsto \sum_{k=1}^{n} f(n) \in \mathcal{C}$
- $n \mapsto \prod_{k=1}^{n} f(n) \in \mathcal{C}$
- $n \mapsto \mathop{\mathrm{K}}_{k=1}^{n} h(n) = h(1) + \cfrac{1}{h(2) + \cfrac{1}{\cdots + \cfrac{1}{h(n)}}} \in \mathcal{C}$

- next week: $\mathcal{C} \subseteq \mathcal{C}^*$, $f \in \mathcal{C}^* \Rightarrow \sqrt[r]{f} \in \mathcal{C}^*$, etc.

Formal definition of $\mathcal{C}$

- Every constant sequence $\{\alpha\}_n$ is in $\mathcal{C}$.
- If $\{f(n)\}_n$ satisfies
$$f(n+r) = p\big(f(n), f(n+1), \ldots, f(n+r-1)\big) \qquad (n \in \mathbb{N})$$
where p is a multivariate polynomial whose coefficients are rational functions of elements in $\mathcal{C}$, then $\{f(n)\}_n \in \mathcal{C}$
- It is assumed that no denominator ever vanishes on the natural numbers.

- Example:
$$f(n+3) = f(n+2)^5 + \frac{n + \mathrm{H}(n)}{\mathrm{F}(n)+1} f(n)f(n+1) + \Big(\prod_{k=1}^{n}(-1)^k\Big)\big(f(n)+f(n+1)+f(n+2)\big)+1$$
$$f(1) = 2, \quad f(2) = 0, \quad f(3) = -\tfrac{2345071}{394757}.$$

- next week: formal definition of $\mathcal{C}^*$.

Algebraic Representation of Elements of $\mathcal{C}$

- **Like Karr(sten):** formal variables t_i represent (sub)expressions
- **Like Burki:** additional formal variables $st_i, sst_i, \ldots$ encode the behaviour of shift.
- Additional variables are also introduced for every denominator.

- Example: $f(n) := \sum_{k=1}^{n} \frac{k!\,\mathrm{H}(k)}{\mathrm{F}(k)}$

$$
\begin{array}{lll}
R_0 := \{st_1 - t_1 - 1, & \sim k & R := \{st_1 - t_1 - 1, sst_1 - st_1 - 1, \\
\quad sst_2 - st_2 - t_2, & \sim \mathrm{F}(k) & \quad sst_2 - st_2 - t_2, \\
\quad st_3 - t_3 st_1, & \sim k! = \Pi k & \quad st_3 - t_3 st_1, sst_3 - st_3 sst_1, \\
\quad t_4 t_1 - 1, & \sim 1/k & \quad t_4 t_1 - 1, st_4 st_1 - 1, sst_4 sst_1 - 1 \\
\quad t_5 t_2 - 1, & \sim 1/\mathrm{F}(k) & \quad t_5 t_2 - 1, st_5 st_2 - 1, sst_5 sst_2 - 1, \\
\quad st_6 - t_6 - st_4, & \sim \mathrm{H}(k) & \quad st_6 - t_6 - st_4, sst_6 - st_6 - sst_4, \\
\quad st_7 - t_7 - st_3 st_5 st_6 \} & \sim f(n) & \quad st_7 - t_7 - st_3 st_5 st_6, sst_7 - st_7 - sst_3 sst_5 sst_6 \}
\end{array}
$$

- For technical convenience, assume R to be filled up till the highest order.

The Proving Algorithm

Input: $f \in \mathcal{C}$ a polynomially recursive function
R defining relations of f and its subexpressions

- Assumption: t_m corresponds to $\{f(n)\}_n$;
The highest order recurrence in R is of order r.

Output: “correct” or “incorrect” depending on whether $f \equiv 0$ or not

function $\mathrm{prove}(f, R)$
  $k := 0$
  $I_0 := \langle R \rangle + \langle t_m, st_m, \ldots, s^{r-1}t_m \rangle \trianglelefteq K[t_*, \ldots, s^r t_*]$
  while $s^{r+k}t_m \notin \mathrm{Rad}\, I_k$
    $k := k + 1$
    $I_k := I_{k-1} + \langle s^{r+k-1}t_m \rangle + \langle s^k R \rangle \trianglelefteq K[t_*, \ldots, s^{k+r}t_*]$
  repeat
  // check initial conditions
  for n **from** 0 **to** $k + r - 1$ **do**
    if $f(n) \neq 0$ **then**
      return “incorrect”, $\mathrm{counterexample} = n$
  return “correct”

Termination

- **Theorem** The algorithm is correct. (obvious)
- **Theorem** The algorithm and terminates.

- **Lemma 1** For every variable $s^r t_i$, there exists exactly one polynomial $p \in R$ with $p = s^r t_i + q$ or $p = q s^r t_i - 1$ for some polynomial q which depends only on variables $s^{r'} t_{i'}$ with $i' \leq i$ and $r' \leq r$, but not on $s^r t_i$.

- **Lemma 2** If X, Y are algebraic sets, Y is irreducible and $X \subsetneq Y$, then $\dim X < \dim Y$.

  Proof Literature, e.g., Theorem I.§6.1.1 in Schafarewitsch 1972

- **Lemma 3** Let $I_k = \langle R \cup sR \cup \cdots \cup s^k R \rangle \trianglelefteq K[t_*, st_*, \ldots, s^{k+r} t_*]$ ($k \in \mathbb{N}$).
  Let $I_n \subseteq \mathfrak{a} \trianglelefteq K[t_*, st_*, \ldots, s^{n+r} t_*]$ for some n be fixed.

  1. $\dim \mathfrak{a} \geq \dim(\mathfrak{a} + I_{n+1})$ where
     the first $\dim$ is wrt. $K[t_*, st_*, \ldots, s^{n+r} t_*]$ and
     the second $\dim$ is wrt. $K[t_*, st_*, \ldots, s^{n+r+1} t_*]$.
  2. $\mathfrak{a}$ has the same number of components as $\mathfrak{a} + I_{n+1}$, or $\mathfrak{a} + I_{n+1} = \langle 1 \rangle$.

Example I

- Consider $\dfrac{\mathrm{F}(n)}{\mathrm{F}(n+1)} + \displaystyle\sum_{k=1}^{n} \dfrac{(-1)^k}{\mathrm{F}(k)\,\mathrm{F}(k+1)} = 0.$

- Put

$$
\begin{aligned}
R := \{ & st_1 + t_1, sst_1 + st_1 && \sim (-1)^n \\
& - sst_2 + st_2 + t_2, && \sim \mathrm{F}(n) \\
& t_3t_2 - 1, st_3st_2 - 1, sst_3sst_2 - 1, && \sim 1/\mathrm{F}(n) \\
& - st_5 + t_5 + t_2st_4t_4, -sst_5 + st_5 + st_2sst_4st_4, && \sim \Sigma \\
& - st_6 + t_3st_4 + st_5, -sst_6 + st_3sst_4 + sst_5 \,\} && \sim f(n)
\end{aligned}
$$

- Write $I_k := \langle R \cup sR \cup \cdots \cup s^k R \rangle$. Loop

$$
\begin{aligned}
I_0 + \langle t_m, st_m, Xsst_m - 1 \rangle &\neq \langle 1 \rangle \\
I_1 + \langle t_m, st_m, sst_m, Xssst_m - 1 \rangle &\neq \langle 1 \rangle \\
I_2 + \langle t_m, \ldots, s^3t_m, Xs^4t_m - 1 \rangle &= \langle 1 \rangle
\end{aligned}
$$

- Check $n = 1, 2, 3, 4$.

Example II

- Consider $f(n) = (n-7)(n-3)\sum_{k=1}^{n}(-1)^k\Big(1+\prod_{k=1}^{n}(-1)^k\Big).$
- We have $f(n) = 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$
- Conjecture: $f \equiv 0$
- Put

$$\begin{aligned} R := \{&-st_1+t_1+1, st_2+t_2, -st_3+t_3+st_2, -st_4+t_4+st_2,\\ &-t_5+(t_1-3)(t_1-7)t_3t_4, -st_5+(st_1-3)(st_1-7)st_3st_4\} \end{aligned}$$

- Loop:

$$\begin{aligned} I_0 + \langle t_5, Xst_5-1\rangle &\neq \langle 1\rangle\\ I_1 + \langle t_5, st_5, Xsst_5-1\rangle &\neq \langle 1\rangle\\ &\vdots\\ I_{10} + \langle t_5, \ldots, s^{10}t_5, Xs^{11}t_5-1\rangle &\neq \langle 1\rangle\\ I_{11} + \langle t_5, \ldots, s^{11}t_5, Xs^{12}t_5-1\rangle &= \langle 1\rangle \end{aligned}$$

- Check 11 initial conditions, find $f(1) = \cdots = f(10) = 0 \neq f(11) = -64$.

Example III

- An identity family with an arbitary function.
- Consider $\displaystyle\sum_{k=1}^{n}\Bigl(\mathrm{H}(k)\sum_{i=1}^{k}f(i)\Bigr) = \bigl(-n+(1+n)\,\mathrm{H}(n)\bigr)\sum_{k=1}^{n}f(k) - \sum_{k=1}^{n}\bigl(\mathrm{H}(k)-1\bigr)kf(k).$
- Let t_4 correspond to $f(n)$. Put

$$\begin{aligned}
R = \{&-st_1 + t_1 + 1, && \sim n\\
&t_2t_1 - 1, st_2st_1 - 1, && \sim 1/n\\
&-st_3 + t_3 + st_2, && \sim \mathrm{H}(n)\\
&-st_5 + t_5 + st_4, && \sim \Sigma f(n)\\
&-st_6 + t_6 + st_1(st_3 - 1)st_4, && \sim \Sigma(\mathrm{H}(n) - 1)nf(n)\\
&-st_7 + t_7 + st_3st_5, && \sim \Sigma(\mathrm{H}(n)\Sigma f(n))\\
&-t_8 - t_7 + (-t_1 + (1 + t_1)t_3)t_5 - t_6,\\
&-st_8 - st_7 + (-st_1 + (1 + st_1)st_3)st_5 - st_6\}
\end{aligned}$$

- $I_0 + \langle t_8, Xst_8 - 1\rangle = \langle 1\rangle$
- Need to check one initial value. $f(1) = f(1)$. QED.

Example IV

- Consider $\displaystyle \mathop{\mathrm{K}}_{k=1}^{n}\left(\frac{1}{4}\frac{(k+1)\,\big((-1)^k(\pi+2)+(\pi-2)\big)\,\Gamma(k/2)}{\sqrt{\pi}\,\Gamma(k/2+1/2)}-\prod_{i=1}^{k}\Big(-\tfrac{7}{4}i^2+9i-\tfrac{45}{4}\Big)\right)=\sum_{k=0}^{n}\frac{1}{k!}$
- Despite some support by the user, Singular, Magma, Asir, and Cocoa run out of memory or compute for a week without any result.
- However, Faugere's Gb system does the job in 8 seconds!
- **Corollary** $\mathrm{e}\notin\mathbb{Q}$ $\displaystyle \left(\ \mathop{\mathrm{K}}_{k=1}^{\infty} a_k\in\mathbb{Q}\overset{?}{\iff} a_n \text{ is periodic}\ \right)$

Correctness and Termination

- **Theorem 1** The algorithm is correct. (obvious)
- **Theorem 2** The algorithm and terminates.

- **Lemma 1** For every variable $s^r t_i$, there exists exactly one polynomial $p \in R$ with $p = s^r t_i + q$ or $p = qs^r t_i - 1$ for some polynomial q which depends only on variables $s^{r'} t_{i'}$ with $i' \leq i$ and $r' \leq r$, but not on $s^r t_i$.

- **Lemma 2** If X, Y are algebraic sets, Y is irreducible and $X \subsetneq Y$, then $\dim X < \dim Y$.

  Proof Literature, e.g., Theorem I.§6.1.1 in Shafarewich 1972

- **Lemma 3** Let $I_k = \langle R \cup sR \cup \cdots \cup s^k R \rangle \trianglelefteq K[t_*, st_*, \ldots, s^{k+r} t_*]$ ($k \in \mathbb{N}$). Let $I_n \subseteq \mathfrak{a} \trianglelefteq K[t_*, st_*, \ldots, s^{n+r} t_*]$ for some n be fixed.
  1. $\dim \mathfrak{a} \geq \dim(\mathfrak{a} + I_{n+1})$ where
     the first $\dim$ is wrt. $K[t_*, st_*, \ldots, s^{n+r} t_*]$ and
     the second $\dim$ is wrt. $K[t_*, st_*, \ldots, s^{n+r+1} t_*]$.
  2. $\mathfrak{a}$ has the same number of components as $\mathfrak{a} + I_{n+1}$, or $\mathfrak{a} + I_{n+1} = \langle 1 \rangle$.

Summary: Proofs of last week

- **Theorem 2** The algorithm terminates.
- Consider primary decomposition of I_k.
- If $s^{k+r}t_m \in \operatorname{Rad} I_k$, then done.
- Otherwise, $s^{k+r}t_m \notin \operatorname{Rad} p$ for at least one primary component.
- $\dim\big(V(s^{k+r}t_m) \cap V(p)\big) < \dim V(p)$.
- Shift does not harm components and their dimension (Lemma 3)
- Dixon's lemma: Replacement of components by others of strictly lower dimension cannot happen infinitely often.
- Hence: At some point, $V(I_k) =$, i.e. $I_k = \langle 1 \rangle$, i.e. $s^{k+r} \in \operatorname{Rad} I_k$.

- **Lemma 3.1** $I_k = \langle R \cup \cdots \cup s^k R\rangle$, $I_{k-1} \subseteq \mathfrak{a} \trianglelefteq K[\ldots, s^{k-1+r}t_m]$. Then $\dim \mathfrak{a} \leq \dim(\mathfrak{a} + I_k)$.
- wlog. $\mathfrak{a}$ primary.
- Extension by one variable increases the dimension by 1
- Show: the relation of the new variable drops the dimension by 1
- Put $X = V(\mathfrak{a} + \langle p \rangle)$, $Y = V(\mathfrak{a})$.
- $X = Y \cap V(p)$.
- Clear: $X \subseteq Y$. Show: $X \subsetneq Y$.
- Let $x = (x_1, \ldots, x_n) \in X$. Then $x' = (x_1, \ldots, x_{n-1}, x_n + 1) \in Y \setminus X$ because p is linear in x_n and cannot have more than one solution.

Work in Progress

- Find interesting examples

  $\rightarrow$ Complete the proof of $\mathrm{e} \notin \mathbb{Q}$

- Find a priori bounds for k.

  $\rightarrow$ Complexity bounds.

  $\rightarrow$ Variants of the algorithm.

- Extension to algebraic functions

- Multivariate sequences

  $\rightarrow$ Closure under definite summation?