The Sum of two Convergent Sequences is Convergent

Manuel Kauers

December 17, 2002

Abstract

This document provides a formal proof of the well known fact that the sum of two convergent sequences itself is convergent, using the tableau method of Smullyan and Fitting.

1 Goal

We want to prove the limit theorem **G** under the assumptions **A1**–**A8** using a tableau calculus for first order predicat logic.

A1 $\underset{\substack{f\\ \mathrm{seq}[f]}}{\forall} (\mathrm{conv}[f] \leftrightarrow \underset{a\in\mathbb{R}}{\exists} \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall} \underset{N\in\mathbb{N}}{\exists} \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f[n]-a| < \varepsilon,$

A2 $\underset{\substack{f\\ \mathrm{seq}[f]}}{\forall} \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall} \mathrm{seq}[f+g],$

A3 $\underset{a\in\mathbb{R}}{\forall} \underset{b\in\mathbb{R}}{\forall} a+b \in \mathbb{R},$

A4 $\underset{\substack{x\in\mathbb{R}\\ x>0}}{\forall} x/2 \in \mathbb{R} \wedge x > 0,$

A5 $\underset{x\in\mathbb{N}}{\forall} \underset{y\in\mathbb{N}}{\forall} \max[x,y] \in \mathbb{N},$

A6 $\underset{x\in\mathbb{N}}{\forall} \underset{y\in\mathbb{N}}{\forall} \underset{z\in\mathbb{N}}{\forall} x \geq \max[y,z] \to x \geq y,$

A7 $\underset{x\in\mathbb{N}}{\forall} \underset{y\in\mathbb{N}}{\forall} \underset{z\in\mathbb{N}}{\forall} x \geq \max[y,z] \to x \geq z,$

A8 $\underset{\substack{f\\ \mathrm{seq}[f]}}{\forall} \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall} \underset{n\in\mathbb{N}}{\forall} \underset{\varepsilon\in\mathbb{R}}{\forall} \underset{a\in\mathbb{R}}{\forall} \underset{b\in\mathbb{R}}{\forall}$

$$(|f[n]-a| < \varepsilon/2 \wedge |g[n]-b| < \varepsilon/2) \to |(f+g)[n] - (a+b)| < \varepsilon$$

G $\underset{\substack{f\\ \mathrm{seq}[f]}}{\forall} \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall} (\mathrm{conv}[f] \wedge \mathrm{conv}[g] \to \mathrm{conv}[f+g]).$

2 Notation

We assume the reader to be familiar with the tableau calculus, and make just some remarks about the notation.

- Each node N of the proof tree has the form

$$n(m) : s\, F$$

  where n is a label of N, m is the label of the node M which justifies N. ($m = \mathbf{G}$ or $m = \mathbf{A}n$ to refer to goal or assumptions), s is the sign and F is the formula of N. (Recall that $s = 0$ means F is assumed to be false and

$s = 1$ means F is assumed to be true). Formulae are denoted in Theorema syntax.

- Free variables are denoted in capital (or greek) letters with asterix superscript, constants are denoted by an index subscript without asterix superscript. All other kinds of variables are bounded by some quantifyer. For instance, X^*, ε^*, and N_2^* are free variables, a_0 and N_1 are constants, and so on.

- Formulae like $\forall_{x \in \mathbb{R}} F(x)$ and $\exists_{x \in \mathbb{R}} F(x)$ are taken as shortcuts for $\forall_x (x \in \mathbb{R} \to F(x))$ and $\exists_x (x \in \mathbb{R} \wedge F(x))$, respectively, and behave accordingly.

- If a path that ends with node n is closed by a substitution σ, say $\sigma = \{X^* \leftarrow x_0\}$, then x_0 might appear instead of X^* in any node with label $m > n$. It is therefore strongly recommended to read subtrees within the proof in the order of the labels rather than in any other order.

- Identifiers of free variables which have been assigned to some term in node n might be recycled in subsequent nodes $m > n$. Also, a variable does not appear as argument of a skolem function, if it has already been assigned to a ground term. (In fact, no skolem function will have arguments due to this rule.)

- Sorry for spoiling the margins, typesetting trees turned out to be a nontrivial task `;-)`

$$0(\mathbf{G})\colon 0 \ \underset{\substack{f\\ \mathrm{seq}[f]}}{\forall}\ \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall}\ (\mathrm{conv}[f] \wedge \mathrm{conv}[g] \to \mathrm{conv}[f+g]).$$

$$1(0)\colon 1\ \mathrm{seq}[f_0].$$

$$2(0)\colon 0 \ \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall}\ (\mathrm{conv}[f_0] \wedge \mathrm{conv}[g] \to \mathrm{conv}[f_0+g]).$$

$$3(2)\colon 1\ \mathrm{seq}[g_0].$$

$$4(2)\colon 0\ \mathrm{conv}[f_0] \wedge \mathrm{conv}[g_0] \to \mathrm{conv}[f_0+g_0].$$

$$5(4)\colon 1\ \mathrm{conv}[f_0] \wedge \mathrm{conv}[g_0].$$

$$6(4)\colon 0\ \mathrm{conv}[f_0+g_0].$$

$$7(5)\colon 1\ \mathrm{conv}[f_0].$$

$$8(5)\colon 1\ \mathrm{conv}[g_0].$$

$$9(\mathbf{A1})\colon 1 \ \underset{\substack{f\\ \mathrm{seq}[f]}}{\forall}\ (\mathrm{conv}[f] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f[n]-a| < \varepsilon).$$

$$10(9)\colon 1\ \mathrm{seq}[F^*] \to (\mathrm{conv}[F^*] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |F^*[n]-a| < \varepsilon).$$

$$11(10)\colon 0\ \mathrm{seq}[F^*].$$

$$\boxed{\textbf{*(11,1)} \text{ by } \{F^* \leftarrow f_0\}}$$

$$12(10)\colon 1\ \mathrm{conv}[f_0] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a| < \varepsilon.$$

$$15(10)\colon 1\ \mathrm{conv}[f_0].$$

$$16(10)\colon 1 \ \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a| < \varepsilon.$$

$$13(10)\colon 0 \ \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a| < \varepsilon.$$

$$14(10)\colon 0\ \mathrm{conv}[f_0].$$

$$\boxed{\textbf{*(7,14)} \text{ by } \{\}}$$

$$17(9)\colon 1\ \mathrm{seq}[G^*] \to (\mathrm{conv}[G^*] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |G^*[n]-a| < \varepsilon).$$

$$19(17)\colon 1\ \mathrm{conv}[G^*] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n]-a| < \varepsilon.$$

$$18(17)\colon 0\ \mathrm{seq}[G^*].$$

$$\boxed{\textbf{*(3,18)} \text{ by } \{G^* \leftarrow g_0\}}$$

$$20(19)\colon 0\ \mathrm{conv}[g_0].$$

$$21(19)\colon 0 \ \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n]-a| < \varepsilon.$$

$$\boxed{\begin{array}{l}\textbf{*(20,8)}\\ \text{by } \{\}\end{array}}$$

$$22(19)\colon 1 \ \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n]-a| < \varepsilon.$$

$$23(19)\colon 1\ \mathrm{conv}[g_0].$$

$$24(9)\colon 1\ \mathrm{seq}[H^*] \to (\mathrm{conv}[H^*] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |H^*[n]-a| < \varepsilon).$$

$$25(24)\colon 1\ \mathrm{conv}[H^*] \leftrightarrow \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |H^*[n]-a| < \varepsilon).$$

$$32(25)\colon 1\ \mathrm{conv}[f_0+g_0].$$

$$33(25)\colon 1 \ \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |(f_0+g_0)[n]-a| < \varepsilon).$$

$$\boxed{\begin{array}{l}\textbf{*(6,32)}\\ \text{by } \{\}\end{array}}$$

$$34(25)\colon 0 \ \underset{a\in\mathbb{R}}{\exists}\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |(f_0+g_0)[n]-a| < \varepsilon).$$

$$35(25)\colon 0\ \mathrm{conv}[f_0+g_0].$$

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$$26(24)\colon 0\ \mathrm{seq}[H^*].$$

$$27(\mathbf{A2})\colon 1 \ \underset{\substack{f\\ \mathrm{seq}[f]}}{\forall}\ \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall}\ \mathrm{seq}[f+g].$$

$$29(27)\colon 1 \ \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall}\ \mathrm{seq}[f_0+g].$$

$$30(29)\colon 0\ \mathrm{seq}[G^{**}].$$

$$\boxed{\begin{array}{l}\textbf{*(3,30)}\\ \text{by } \{G^{**} \leftarrow g_0\}\end{array}}$$

$$31(29)\colon 1\ \mathrm{seq}[f_0+g_0].$$

$$\boxed{\begin{array}{l}\textbf{*(26,31)}\\ \text{by } \{H^* \leftarrow f_0+g_0\}\end{array}}$$

$$28(27)\colon 0\ \mathrm{seq}[F^{**}].$$

$$\boxed{\begin{array}{l}\textbf{*(1,28)}\\ \text{by } \{F^{**} \leftarrow f_0\}\end{array}}$$

$\vdots$

$35(25)\colon 0\ \mathrm{conv}[f_0+g_0].$

$36(16)\colon 1\ a_0 \in \mathbb{R} \wedge (\underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a_0| < \varepsilon).$

$37(36)\colon 1\ a_0 \in \mathbb{R}.$

$38(36)\colon 1\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a_0| < \varepsilon.$

$39(22)\colon 1\ b_0 \in \mathbb{R} \wedge (\underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n]-b_0| < \varepsilon).$

$40(39)\colon 1\ b_0 \in \mathbb{R}.$

$41(39)\colon 1\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n]-b_0| < \varepsilon.$

$42(34)\colon 0\ C^* \in \mathbb{R} \wedge (\underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |(f_0+g_0)[n]-C^*| < \varepsilon).$

$51(42)\colon 0\ \underset{\substack{\varepsilon\in\mathbb{R}\\ \varepsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |(f_0+g_0)[n]-(a_0+b_0)| < \varepsilon.$

$52(51)\colon 0\ (\varepsilon_0 \in \mathbb{R} \wedge \varepsilon_0 > 0) \to \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |(f_0+g_0)[n]-(a_0+b_0)| < \varepsilon_0.$

$53(52)\colon 1\ \varepsilon_0 \in \mathbb{R} \wedge \varepsilon_0 > 0.$

$54(52)\colon 0\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |(f_0+g_0)[n]-(a_0+b_0)| < \varepsilon_0.$

$55(53)\colon 1\ \varepsilon_0 \in \mathbb{R}.$

$56(53)\colon 1\ \varepsilon_0 > 0.$

$57(38)\colon 1\ (E_1^* \in \mathbb{R} \wedge E_1^* > 0) \to \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a_0| < E_1^*.$

$58(\mathbf{A4})\colon 1\ \underset{\substack{x\in\mathbb{R}\\ x>0}}{\forall} x/2 \in \mathbb{R} \wedge x/2 > 0.$

$69(57)\colon 1\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |f_0[n]-a_0| < \varepsilon_0/2.$

$70(41)\colon 1\ (E_2^* \in \mathbb{R} \wedge E_2^* > 0) \to \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n]-b_0| < E_2^*).$

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$59(57)\colon 0\ E_1^* \in \mathbb{R} \wedge E_1^* > 0.$

$60(58)\colon 1\ (X^* \in \mathbb{R} \wedge X^* > 0) \to (X^*/2 \in \mathbb{R} \wedge X^*/2 > 0).$

$64(60)\colon 1\ \varepsilon_0/2 \in \mathbb{R} \wedge \varepsilon_0/2 > 0.$

$65(64)\colon 1\ \varepsilon_0/2 \in \mathbb{R}.$

$66(64)\colon 1\ \varepsilon_0/2 > 0.$

$67(59)\colon 0\ E_1^* \in \mathbb{R}.$

***(65,67)** by $\{E_1^* \leftarrow \varepsilon_0/2\}$

$68(59)\colon 0\ \varepsilon_0/2 > 0.$

***(66,68)** by $\{\}$

$61(60)\colon 0\ X^* \in \mathbb{R} \wedge X^* > 0.$

$62(61)\colon 0\ X^* \in \mathbb{R}.$

***(55,62)** by $\{X^* \leftarrow \varepsilon_0\}$

$63(61)\colon 0\ \varepsilon_0 > 0.$

***(56,63)** by $\{\}$

$43(42)\colon 0\ C^* \in \mathbb{R}.$

$44(\mathbf{A3})\colon 1\ \underset{a\in\mathbb{R}}{\forall}\ \underset{b\in\mathbb{R}}{\forall} a+b \in \mathbb{R}.$

$45(44)\colon 1\ A^* \in \mathbb{R} \to \underset{b\in\mathbb{R}}{\forall} A^*+b \in \mathbb{R}.$

$46(45)\colon 0\ A^* \in \mathbb{R}.$

***(37,46)** by $\{A^* \leftarrow a_0\}$

$47(45)\colon 1\ \underset{b\in\mathbb{R}}{\forall} a_0+b \in \mathbb{R}.$

$48(47)\colon 1\ B^* \in \mathbb{R} \to a_0+B^* \in \mathbb{R}.$

$49(48)\colon 0\ B^* \in \mathbb{R}.$

***(40,49)** by $\{B^* \leftarrow b_0\}$

$50(48)\colon 1\ a_0+b_0 \in \mathbb{R}.$

***(43,50)** by $\{C^* \leftarrow a_0+b_0\}$

$\vdots$

$70(41){:}\ 1\ (E_2^* \in \mathbb{R} \wedge E_2^* > 0) \rightarrow \underset{N\in\mathbb{N}}{\exists} \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n] - b_0| < E_2^*).$

$71(70){:}\ 0\ E_2^* \in \mathbb{R} \wedge E_2^* > 0.$ $\quad 81(70){:}\ 1 \underset{N\in\mathbb{N}}{\exists} \underset{\substack{n\in\mathbb{N}\\ n\geq N}}{\forall} |g_0[n] - b_0| < E_2^*.$

$72(58){:}\ 1\ (X^* \in \mathbb{R} \wedge X^* > 0) \rightarrow (X^*/2 \in \mathbb{R} \wedge X^*/2 > 0).$ $\quad 82(69){:}\ 1\ N_1 \in \mathbb{N} \wedge \underset{\substack{n\in\mathbb{N}\\ n\geq N_1}}{\forall} |f_0[n] - a_0| < \varepsilon_0/2.$

$76(72){:}\ 1\ \varepsilon_0/2 \in \mathbb{R} \wedge \varepsilon_0/2 > 0.$ $\quad 73(72){:}\ 0\ X^* \in \mathbb{R} \wedge X^* > 0.$ $\quad 83(82){:}\ 1\ N_1 \in \mathbb{N}.$

$77(76){:}\ 1\ \varepsilon_0/2 \in \mathbb{R}.$ $\quad 74(73){:}\ 0\ X^* \in \mathbb{R}.$ $\quad 75(73){:}\ 0\ \varepsilon_0 > 0.$ $\quad 84(82){:}\ 1 \underset{\substack{n\in\mathbb{N}\\ n\geq N_1}}{\forall} |f_0[n] - a_0| < \varepsilon_0/2.$

***(55,74)** by $\{X^* \leftarrow \varepsilon_0\}$ $\quad$ ***(56,75)** by $\{\}$

$78(76){:}\ 1\ \varepsilon_0/2 > 0.$ $\quad 85(81){:}\ 1\ N_2 \in \mathbb{N} \wedge \underset{\substack{n\in\mathbb{N}\\ n\geq N_2}}{\forall} |g_0[n] - b_0| < \varepsilon_0/2.$

$79(71){:}\ 0\ E_2^* \in \mathbb{R}.$ $\quad 80(71){:}\ 0\ \varepsilon_0/2 > 0.$ $\quad 86(85){:}\ 1\ N_2 \in \mathbb{N}.$

***(77,79)** by $\{E_2^* \leftarrow \varepsilon_0/2\}$ $\quad$ ***(78,80)** by $\{\}$

$87(85){:}\ 1 \underset{\substack{n\in\mathbb{N}\\ n\geq N_2}}{\forall} |g_0[n] - b_0| < \varepsilon_0/2.$

$88(54){:}\ 0\ N^* \in \mathbb{N} \wedge \underset{\substack{n\in\mathbb{N}\\ n\geq N^*}}{\forall} |(f_0 + g_0)[n] - (a_0 + b_0)| < \varepsilon_0.$

$97(88){:}\ 0 \underset{\substack{n\in\mathbb{N}\\ n\geq \max[N_1, N_2]}}{\forall} |(f_0 + g_0)[n] - (a_0 + b_0)| < \varepsilon_0.$ $\quad 89(88){:}\ 0\ N^* \in \mathbb{N}.$

$98(97){:}\ 0\ (n_0 \in \mathbb{N} \wedge n_0 \geq \max[N_1, N_2]) \rightarrow |(f_0 + g_0)[n_0] - (a_0 + b_0)| < \varepsilon_0.$ $\quad 90(\mathbf{A5}){:}\ 1 \underset{x\in\mathbb{N}}{\forall} \underset{y\in\mathbb{N}}{\forall} \max[x, y] \in \mathbb{N}.$

$99(98){:}\ 1\ n_0 \in \mathbb{N} \wedge n_0 \geq \max[N_1, N_2].$ $\quad 91(90){:}\ 1\ X^* \in \mathbb{N} \rightarrow \underset{y\in\mathbb{N}}{\forall} \max[X^*, y] \in \mathbb{N}.$

$100(98){:}\ 0\ |(f_0 + g_0)[n_0] - (a_0 + b_0)| < \varepsilon_0.$ $\quad 93(91){:}\ 1 \underset{y\in\mathbb{N}}{\forall} \max[N_1, y] \in \mathbb{N}.$ $\quad 92(91){:}\ 0\ X^* \in \mathbb{N}.$

***(83,92)** by $\{X^* \leftarrow N_1\}$

$101(99){:}\ 1\ n_0 \in \mathbb{N}.$ $\quad 94(93){:}\ 1\ Y^* \in \mathbb{N} \rightarrow \max[N_1, Y^*] \in \mathbb{N}.$

$102(99){:}\ 1\ n_0 \geq \max[N_1, N_2].$ $\quad 95(94){:}\ 0\ Y^* \in \mathbb{N}.$ $\quad 96(94){:}\ 1\ \max[N_1, N_2] \in \mathbb{N}.$

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***(86,95)** by $\{Y^* \leftarrow N_2\}$ $\quad$ ***(89,96)** by $\{N^* \leftarrow \max[N_1, N_2]\}$

$\vdots$

102(99): 1 $n_0 \geq \max[N_1, N_2]$.

103(84): 1 $(N_1^* \in \mathbb{N} \wedge N_1^* \geq N_1) \rightarrow |f_0[N_1^*] - a_0| < \varepsilon_0/2$.

104(103): 0 $N_1^* \in \mathbb{N} \wedge N_1^* \geq N_1$.   116(103): 1 $|f_0[n_0] - a_0| < \varepsilon_0/2$.

105(104): 0 $N_1^* \in \mathbb{N}$.   106(104): 0 $n_0 \geq N_1$.   **next tree**

***(101,105)** by $\{N_1^* \leftarrow n_0\}$ 107(**A6**): 1 $\underset{x\in\mathbb{N}}{\forall}\underset{y\in\mathbb{N}}{\forall}\underset{z\in\mathbb{N}}{\forall} x \geq \max[y,z] \rightarrow x \geq y$.

108(107): 1 $(X^* \in \mathbb{N} \wedge Y^* \in \mathbb{N} \wedge Z^* \in \mathbb{N}) \rightarrow (X^* \geq \max[Y^*, Z^*] \rightarrow X^* \geq Y^*)$.

109(108): 1 $X^* \geq \max[Y^*, Z^*] \rightarrow X^* \geq Y^*$.   112(108): 0 $n_0 \in \mathbb{N} \wedge N_1 \in \mathbb{N} \wedge N_2 \in \mathbb{N}$.

110(109): 0 $X^* \geq \max[Y^*, Z^*]$.   111(109): 1 $n_0 \geq N_1$.   113(112): 0 $n_0 \in \mathbb{N}$.   114(112): 0 $N_1 \in \mathbb{N}$.   115(112): 0 $N_2 \in \mathbb{N}$.

***(102,110)** by $\{X^* \leftarrow n_0, Y^* \leftarrow N_1, Z^* \leftarrow N_2\}$ ***(106,111)** by $\{\}$ ***(101,113)** by $\{\}$ ***(83,114)** by $\{\}$ ***(86,115)** by $\{\}$

$\vdots$

116(103): 1 $|f_0[n_0] - a_0| < \varepsilon_0/2$.

117(87): 1 $(N_2^* \in \mathbb{N} \wedge N_2^* \geq N_2) \rightarrow |g_0[N_2^*] - b_0| < \varepsilon_0/2$.

118(117): 0 $N_2^* \in \mathbb{N} \wedge N_2^* \geq N_2$.   130(117): 1 $|g_0[n_0] - b_0| < \varepsilon_0/2$.

119(118): 0 $N_2^* \in \mathbb{N}$.   120(118): 0 $n_0 \geq N_2$.   **next tree**

***(101,119)** by $\{N_2^* \leftarrow n_0\}$ 121(**A7**): 1 $\underset{x\in\mathbb{N}}{\forall}\underset{y\in\mathbb{N}}{\forall}\underset{z\in\mathbb{N}}{\forall} x \geq \max[y,z] \rightarrow x \geq z$.

122(121): 1 $(X^* \in \mathbb{N} \wedge Y^* \in \mathbb{N} \wedge Z^* \in \mathbb{N}) \rightarrow (X^* \geq \max[Y^*, Z^*] \rightarrow X^* \geq Z^*)$.

123(122): 1 $X^* \geq \max[Y^*, Z^*] \rightarrow X^* \geq Z^*$.   126(122): 0 $n_0 \in \mathbb{N} \wedge N_1 \in \mathbb{N} \wedge N_2 \in \mathbb{N}$.

124(123): 0 $X^* \geq \max[Y^*, Z^*]$.   125(123): 1 $n_0 \geq N_2$.   127(126): 0 $n_0 \in \mathbb{N}$.   128(126): 0 $N_1 \in \mathbb{N}$.   129(126): 0 $N_2 \in \mathbb{N}$.

***(102,124)** by $\{X^* \leftarrow n_0, Y^* \leftarrow N_1, Z^* \leftarrow N_2\}$ ***(120,125)** by $\{\}$ ***(101,127)** by $\{\}$ ***(83,128)** by $\{\}$ ***(86,129)** by $\{\}$

$\vdots$

130(117): 1 $|g_0[n_0] - b_0| < \varepsilon_0/2.$

131(**A8**): 1 $\underset{\substack{f\\ \mathrm{seq}[f]}}{\forall}\ \underset{\substack{g\\ \mathrm{seq}[g]}}{\forall}\ \underset{n\in\mathbb{N}}{\forall}\ \underset{\varepsilon\in\mathbb{R}}{\forall}\ \underset{a\in\mathbb{R}}{\forall}\ \underset{b\in\mathbb{R}}{\forall}(|f[n] - a| < \varepsilon/2 \wedge |g[n] - b| < \varepsilon/2) \rightarrow |(f+g)[n] - (a+b)| < \varepsilon.$

132(131): 1 $(\mathrm{seq}[F^{**}], \mathrm{seq}[G^{**}], N^{**} \in \mathbb{N}, E^{**} \in \mathbb{R}, A^{**} \in \mathbb{R}, B^{**} \in \mathbb{R}) \rightarrow (|f[n] - a| < \varepsilon/2 \wedge |g[n] - b| < \varepsilon/2) \rightarrow |(f+g)[n] - (a+b)| < \varepsilon.$

133(132): 0 $\mathrm{seq}[F^{**}], \mathrm{seq}[G^{**}], N^{**} \in \mathbb{N}, E^{**} \in \mathbb{R}, A^{**} \in \mathbb{R}, B^{**} \in \mathbb{R}.$

134(133): 0 $\mathrm{seq}[F^{**}].$

***(1,134)** by $\{F^{**} \leftarrow f_0\}$

135(133): 0 $\mathrm{seq}[G^{**}].$

***(3,135)** by $\{G^{**} \leftarrow g_0\}$

136(133): 0 $N^{**} \in \mathbb{N}.$

***(101,136)** by $\{N^{**} \leftarrow n_0\}$

137(133): 0 $E^{**} \in \mathbb{R}.$

***(55,137)** by $\{E^{**} \leftarrow \varepsilon_0\}$

138(133): 0 $A^{**} \in \mathbb{R}.$

***(37,138)** by $\{A^{**} \leftarrow a_0\}$

139(133): 0 $B^{**} \in \mathbb{R}.$

***(40,139)** by $\{B^{**} \leftarrow b_0\}$

140(132): 1 $(|f_0[n_0] - a_0| < \varepsilon_0/2 \wedge |g_0[n_0] - b_0| < \varepsilon_0/2) \rightarrow |(f_0 + g_0)[n_0] - (a_0 + b_0)| < \varepsilon_0.$

141(140): 0 $|f_0[n_0] - a_0| < \varepsilon_0/2 \wedge |g_0[n_0] - b_0| < \varepsilon_0/2.$

142(141): 0 $|f_0[n_0] - a_0| < \varepsilon_0/2.$

***(116,142)** by {}

143(141): 0 $|g_0[n_0] - b_0| < \varepsilon_0/2.$

***(130,143)** by {}

144(140): 1 $|(f_0 + g_0)[n_0] - (a_0 + b_0)| < \varepsilon_0.$

***(100,144)** by {}