The Composition of Continuous Functions is Continuous

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TSW-Homework

A function f is called *continuous* if

$$\underset{y}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x}{\forall} |x-y|<\delta \to |f(x)-f(y)|<\varepsilon. \tag{1}$$

We claim that if f and g are continuous then $f \circ g \colon x \mapsto f(g(x))$ is continuous as well.

Proof: Assume that f and g are continuous, i.e.

$$\underset{y}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x}{\forall} |x-y|<\delta \to |f(x)-f(y)|<\varepsilon \tag{A1}$$

$$\underset{y}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x}{\forall} |x-y|<\delta \to |g(x)-g(y)|<\varepsilon \tag{A2}$$

We want to show

$$\underset{y}{\forall}\ \underset{\varepsilon>0}{\forall}\ \underset{\delta}{\exists}\ \underset{x}{\forall} |x-y|<\delta \to |f(g(x))-f(g(y))|<\varepsilon \tag{G1}$$

For arbitrary but fixed y_0, ε_0 with

$$\varepsilon_0 > 0, \tag{A3}$$

we have to show

$$\underset{\delta>0}{\exists}\ \underset{x}{\forall} |x-y_0|<\delta \to |f(g(x))-f(g(y_0))|<\varepsilon_0. \tag{G2}$$

In (A1) we can take $y = g(y_0)$ and $\varepsilon = \varepsilon_0$ (using (A3)) obtaining

$$\underset{\delta>0}{\exists}\ \underset{x}{\forall} |x-g(y_0)|<\delta \to |f(x)-f(g(y_0))|<\varepsilon_0. \tag{A4}$$

Now we take such a δ_0, i.e. assume

$$\delta_0 > 0, \tag{A5}$$

$$\underset{x}{\forall} |x-g(y_0)|<\delta_0 \to |f(x)-f(g(y_0))|<\varepsilon_0. \tag{A6}$$

For this δ_0 and for $y = y_0$ we know from (A2) and (A5) that

$$\underset{\delta>0}{\exists}\ \underset{x}{\forall} |x-y_0|<\delta \to |g(x)-g(y_0)|<\delta_0. \tag{A7}$$

Fix such a δ and call it δ_1, then we know that

$$\delta_1 > 0, \tag{A8}$$

$$\underset{x}{\forall} |x - y_0| < \delta_1 \rightarrow |g(x) - g(y_0)| < \delta_0. \tag{A9}$$

We will show that δ_1 manages to play the role of δ in (G2) by showing

$$\underset{x}{\forall} |x - y_0| < \delta_1 \rightarrow |f(g(x)) - f(g(y_0))| < \varepsilon_0. \tag{G3}$$

To show (G3), we show

$$|x_0 - y_0| < \delta_1 \rightarrow |f(g(x_0)) - f(g(y_0))| < \varepsilon_0 \tag{G4}$$

for an arbitrary but fixed x_0. Note that $\delta_0 > 0$ by (A8). By (A9) we know that

$$|x_0 - y_0| < \delta_1 \rightarrow |g(x_0) - g(y_0)| < \delta_0 \tag{A10}$$

and taking $x = g(x_0)$ in (A6) yields

$$|g(x_0) - g(y_0)| < \delta_0 \rightarrow |f(g(x_0)) - f(g(y_0))| < \varepsilon_0. \tag{A11}$$

Finally, by the transitivity of implication, (A10) and (A11) give

$$|x_0 - y_0| < \delta_1 \rightarrow |f(g(x_0)) - f(g(y_0))| < \varepsilon_0 \tag{A12}$$

quod erat demonstrandum.