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Algebraic Topology

Summary to a Lecture of Günther Landsmann

Contents

1 Categories

For the presentation of Algebraic Topology which is planed to be given in this course, we want to use tools from category theory. This first section will provide the necessary background about categories.

Definition 1 Let $\mathfrak{C}$ consist of a class C of *objects*, of a set $M(X,Y) = M_{\mathfrak{C}}(X,Y)$ of *morphisms* for any $X, Y \in C$, and of a map $\circ\colon M(X,Y) \times M(Y,Z) \to M(X,Z)$ for any three objects X, Y, Z. $\mathfrak{C}$ is called a *category*, if

1. for any $X, Y, Z, U \in C$ and any $f \in M(X,Y)$, $g \in M(Y,Z)$, $h \in M(Z,U)$, we have $f \circ (g \circ h) = (f \circ g) \circ h$, and
2. for any $X \in C$ there is a morphism $1 = 1_X \in M(X,X)$ with the property $1_X \circ f = f$ $(f \in M(Y,X))$ and $g \circ 1_X = g$ $(g \in M(X,Z))$.

Instead of $X \in C$ we will also write $X \in \mathfrak{C}$. Also, we will write $\mathfrak{C} = (C, M, \circ)$ to indicate that C be the class of objects, $M(X,Y)$ be the set of morphisms from X to Y, and that $\circ$ be the composition function of morphisms.

In the present context, the class C will usually contain sets, and $M(X,Y)$ will be a set of certain maps from X to Y. In this setting, when taking for $\circ$ the composition of maps, it is clear that C, M, and $\circ$ fulfill the two axioms and hence build a category. Categories whose objects are sets and where the morphisms are maps between the sets are called *concrete categories*. If the objects of a category fit into a set (rather than to a class), then we speak about a *small category*.

The easiest category we are interested in is $\mathfrak{Set}$, the category of sets. Its objects are all sets (note that the objects need not belong a set, but only to a "class"), and $M(X,Y)$ is the set of all maps from X to Y. In fact, many mathematical structurs can be viewed as categories: The category $\mathfrak{Group}$ of groups (with homomorphisms as morphisms), the category $\mathfrak{VecSpace}$ of vector spaces (with linear maps as morphisms), and so on.

The idea behind the notion of a category is that many similar questions that arise in different mathematical structures may be treated once and for all in a more abstract setting. For instance, the notion of *isomorphism* between objects can be defined on the level of categories.

Definition 2 Let $\mathfrak{C}$ be a category and X, Y be objects of $\mathfrak{C}$. These objects are called *isomorphic* if there are morphisms $f \in M(X,Y)$ and $g \in M(Y,X)$ such that $1_X = g \circ f$ and $1_Y = f \circ g$. Such f and g are called *isomorphisms*, and we write $X \cong Y$ if X and Y are isomorphic.

Clearly, $X, Y \in \mathfrak{Set}$ are isomorphic iff they have the same cardinality, $X, Y \in \mathfrak{Group}$ are isomorphic iff they are isomorphic in the usual sense, and so on. It is also easy to see that isomorphism is an equivalence relation on a category.

In the next definition, we consider the relationship between two categories.

Definition 3 Let $\mathfrak{A}, \mathfrak{B}$ be categories.

1. A *covariant functor* $F\colon \mathfrak{A} \to \mathfrak{B}$ is a mapping that maps objects $X \in \mathfrak{A}$ to objects $F(X) \in \mathfrak{B}$ and morphisms $m \in M_{\mathfrak{A}}(X,Y)$ to morphisms $F(m) \in M_{\mathfrak{B}}(F(X), F(Y))$ such that $F(1_X) = 1_{F(X)}$ for all $X \in \mathfrak{A}$ and $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ for all $\varphi \in M_{\mathfrak{A}}(X,Y)$, $\psi \in M_{\mathfrak{A}}(Y,Z)$.

2. A *contravariant functor* $F\colon \mathfrak{A} \to \mathfrak{B}$ is a mapping that maps objects $X \in \mathfrak{A}$ to objects $F(X) \in \mathfrak{B}$ and morphisms $m \in M_{\mathfrak{A}}(X,Y)$ to morphisms $F(m) \in M_{\mathfrak{B}}(F(Y), F(X))$ such that $F(1_X) = 1_{F(X)}$ for all $X \in \mathfrak{A}$ and $F(\psi \circ \varphi) = F(\varphi) \circ F(\psi)$ for all $\varphi \in M_{\mathfrak{A}}(X,Y)$, $\psi \in M_{\mathfrak{A}}(Y,Z)$.

The difference between covariant and contravariant functor is that the contravariant functor also changes the direction of the morphisms, i.e., if φ is a morphism from X to Y, then $F(\varphi)$ is a morphism from $F(X)$ to $F(Y)$ in case F is covariant, and $F(\varphi)$ is a morphism from $F(Y)$ to $F(X)$ in case F is contravariant.

Example 1 The *forgetful functor* $|\cdot|\colon \mathfrak{Group} \to \mathfrak{Set}$ maps a group G to its underlying set $|G|$ and homomorphisms to the corresponding maps — it "forgets" the group structure. The forgetful functor can be analogously defined for other categories $\mathfrak{A}$ in place of $\mathfrak{Group}$ as long as the objects of $\mathfrak{A}$ are sets with a certain structure.

The distinction between covariant and contravariant functors is not essential, but convenient. In fact, contravariant functors from a category $\mathfrak{A}$ to a category $\mathfrak{B}$ can also be viewed as covariant functors from $\mathfrak{A}$ to the *dual category* $\mathfrak{B}^*$ which is defined as follows.

Definition 4 Let $\mathfrak{C}$ be a category with objects in a class C, where $M(X,Y)$ defines the morphisms from X to Y and $\circ\colon M(X,Y) \times M(Y,Z) \to M(X,Z)$ the composition of morphisms. Further, let $\mathfrak{C}^*$ be the category which consists of the same objects as $\mathfrak{C}$ does, but with $M^*(X,Y) := M(Y,X)$ as the set of morphisms and with $f \circ^* g := g \circ f$ as composition function. Then $\mathfrak{C}^*$ is called the *dual category* of $\mathfrak{C}$.

We conclude this section by another example of a functor which will be used frequently in the sequel.

Example 2 Let $\mathfrak{C} = (C, M, \circ)$ be a category, $A \in \mathfrak{C}$ and $A^* \in \mathfrak{C}^*$. The functors

$$\begin{aligned} &\mathrm{hom}_{\mathfrak{C},A^*}\colon \mathfrak{C} \to \mathfrak{Set}, &&\mathrm{hom}_{\mathfrak{C},A^*}(X) := M_{\mathfrak{C}}(A^*, X), \\ &\mathrm{hom}_{\mathfrak{C}^*,A}\colon \mathfrak{C}^* \to \mathfrak{Set}, &&\mathrm{hom}_{\mathfrak{C}^*,A}(X) := M_{\mathfrak{C}}(X, A) \end{aligned}$$

are called the *covariant* hom *functor* and the *contravariant* hom *functor* of $\mathfrak{C}$ with respect to A and A^*, respetively.
A morphism $m \in M_{\mathfrak{C}}(X,Y)$ is mapped by $\mathrm{hom}_{\mathfrak{C},A^*}$ to the morphism

$$\mathrm{hom}_{\mathfrak{C},A^*}(m)\colon M_{\mathfrak{C}}(A^*, X) \to M_{\mathfrak{C}}(A^*, Y), \qquad \mathrm{hom}_{\mathfrak{C},A^*}(m)(\varphi) := m \circ \varphi.$$

The morphism map of the contravariant functor $\mathrm{hom}_{\mathfrak{C},A}$ is defined analogously.

The "combination" of both, covariant and contravariant hom functor,

$$\hom := \hom_{\mathfrak{C}} \colon \mathfrak{C}^* \times \mathfrak{C} \to \mathfrak{Set}, \qquad \hom(X,Y) := M_{\mathfrak{C}}(X,Y)$$

is simply called the hom *functor* of $\mathfrak{C}$. This functor maps morphisms $(m_1, m_2) \in M_{\mathfrak{C}^*}(X_1, Y_1) \times M_{\mathfrak{C}}(X_2, Y_2)$ to

$$\hom(m_1, m_2) \colon M_{\mathfrak{C}}(X_1, X_2) \to M_{\mathfrak{C}}(Y_1, Y_2), \quad \hom(m_1, m_2)(\varphi) := m_2 \circ \varphi \circ m_1.$$

2 Topology

2.1 The Category of Spaces

We turn to the objects of our interest, the topological spaces. By $\mathfrak{Top}$ we denote the category of spaces. An object $X \in \mathfrak{Top}$ is a topological space, that is, a set $|X|$ together with a system $\mathcal{T} \subseteq \mathcal{P}(|X|)$ of *open sets*. The morphisms $\varphi \in M_{\mathfrak{Top}}(X,Y)$ are the continous functions form X to Y; we will write $C(X,Y)$ instead of $M_{\mathfrak{Top}}(X,Y)$.

2.1.1 Products and Coproducts

The category $\mathfrak{Top}$ is closed under taking products and coproducts. If X_i $(i \in I)$ are topological spaces then so are $\prod_{i \in I} X_i$ and $\coprod_{i \in I} X_i$. By π_j or π_{X_j} we will denote the *projection* from $\prod_{i \in I} X_i$ to X_j. By i_j we denote the *injection* from X_j to $\coprod_{i \in I} X_i$.

Theorem 1 Let X_i $(i \in I)$ and Y be spaces.

1. Let $f_i \in C(Y, X_i)$ for every $i \in I$. Then there exists a unique morphism $h \in C(Y, \prod_{i \in I} X_i)$ such that $f_j = h \circ \pi_j$ for every $j \in I$.
2. Let $f_i \in C(X_i, Y)$ for every $i \in X_i$. Then there exists a unique morphism $h \in C(\coprod_{i \in I} X_i, Y)$ such that $f_j = i_j \circ h$ for every $j \in I$.

The situations are illustrated in the following diagrams.

$$\begin{tikzcd}
 & & X_1 \\
Y \arrow[urr, "f_1"] \arrow[r, "h"] \arrow[drr, "f_n"'] & \prod_{i \in I} X_i \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_n"] & \vdots \\
 & & X_n
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[dr, "i_1"'] \arrow[drr, "f_1"] & & \\
\vdots & \coprod_{i \in I} X_i \arrow[r, "h"] & Y \\
X_n \arrow[ur, "i_n"] \arrow[urr, "f_n"'] & &
\end{tikzcd}$$

Theorem 1 might be taken as the defining property of product and coproduct in the following sense. The product topology $\prod_{i\in I} X_i$ of X_i $(i \in I)$ consists of the set theoretic cartesian product of the sets $|X_i|$ $(i \in I)$ equipped with the initial topology that makes the left diagram commute. Analogously, the coproduct $\coprod_{i\in I} X_i$ consists of the set

$$|\coprod_{i\in I} X_i| = \bigcup_{i\in I}(X_i \times \{i\})$$

equipped with the final topology that makes the right diagram commute.

2.1.2 Internalization

Given a category $\mathfrak{C}$ whose objects are sets with a certain structure, we want to equip the set $\hom(X, Y)$ also with such a structure, i.e., we want it to become an object of $\mathfrak{C}$. This is called *internalization*.

At present, we deal with the category $\mathfrak{Top}$ of topological spaces. For $X, Y \in \mathfrak{Top}$, we have $C(X, Y)$ as the set of morphisms. In order to get this set itself a member of $\mathfrak{Top}$, we have to equip it with a topology. It will turn out that the following topology is suitable.

Definition 5 Let $X, Y \in \mathfrak{Top}$. The *compact open topology* on $C(X, Y)$ is defined to be the topology generated by

$$\{f \in C(X, Y) : \forall\ K \subseteq X \text{ compact}, U \subseteq Y \text{ open} : f(K) \subseteq U\}$$

as a subbase. We will write X^Y for the space $C(X, Y)$ with the compact open topology.

This is of course not the only way to define a topology on $C(X, Y)$. But the compact open topology has the property that it "factors through $\mathfrak{Top}$." More formally, this means the following.

Theorem 2 Let the functor Hom be defined by

$$\mathrm{Hom}\colon \mathfrak{Top}^* \times \mathfrak{Top} \to \mathfrak{Top}, \qquad \mathrm{Hom}(X, Y) := X^Y.$$

Then $\mathrm{Hom} \circ |\cdot| = \hom$ where hom is the functor of Example 2 and $|\cdot|$ is the forgetful functor defined analogously to Example 1.

$$\begin{tikzcd}
\mathfrak{Top}^* \times \mathfrak{Top} \arrow[rr, "\mathrm{hom}"] \arrow[dr, "\mathrm{Hom}"'] & & \mathfrak{Set} \\
& \mathfrak{Top} \arrow[ur, "|\cdot|"'] &
\end{tikzcd}$$

The statement of the theorem above is clear for the objects of the involved categories, but it also has to hold for the morphisms between the objects. In particular, it states that $\mathrm{Hom}(f_1, f_2)\colon C(X_1, X_2) \to C(Y_1, Y_2)$ is continous with respect to compact open topology, if $f_1 \in C(Y_1, X_1)$ and $f_2 \in C(X_2, Y_2)$.

Definition 6 Let $X, Y \in \mathfrak{Top}$. A topology on $C(X,Y)$ is called *admissible* if the evaluation function

$$e\colon C(X,Y) \times X \to Y, \qquad e(f,x) := f(x)$$

is continous.

There are spaces X, Y where no admissible topology on $C(X,Y)$ in the sense of the above definition exists. But in case there exist admissible topologies, their are related to the compact open topology as follows

Theorem 3 Let $X, Y \in \mathfrak{Top}$.

- Every admissible topology on $C(X,Y)$ contains the compact open topology.
- If X is locally compact T_2 then the compact open topology on $C(X,Y)$ is admissible.
- If Y is locally compact T_2 then the composition $\circ\colon Z^Y \times Y^X \to Y^X$ is continous.

Putting things together, we get that the compact open topology is the croasest admissibile topology on $C(X,Y)$ if X is locally compact T_2.

Theorem 4 (Exponential Law in $\mathfrak{Top}$) Let $X, Y, Z \in \mathfrak{Top}$, X Hausdorff, and Y locally compact T_2. Then

$$Z^{X \times Y} \cong (Z^X)^Y$$

in $\mathfrak{Top}$.

2.2 The Category of Pairs

We consider pairs (X, A) of topological spaces where $A \subseteq X$ is a subspace of X. The *category of pairs* is denoted by $\mathfrak{Top}_2$. Its objects are the pairs (X, A), and the morphisms are naturally defined by

$$M((X,A),(Y,B)) := \hom((X,A),(Y,B)) := \{f \in C(X,Y) : f(A) \subseteq B\}.$$

It is clear that $\mathfrak{Top}$ is a subcategory of $\mathfrak{Top}_2$ by writing spaces $X \in \mathfrak{Top}$ in the form $(X, \emptyset) \in \mathfrak{Top}_2$.
We will find that $\mathfrak{Top}_2$ behaves similar to $\mathfrak{Top}$. If we define

$$\prod_{i \in I}(X_i, A_i) := \Big(\prod_{i \in I} X_i, \prod_{i \in I} A_i\Big), \qquad \coprod_{i \in I}(X_i, A_i) := \Big(\coprod_{i \in I} X_i, \coprod_{i \in I} A_i\Big),$$

we get in analogy to Theorem 1 the following result:

Theorem 5 Let (X_i, A_i) $(i \in I)$ and (Y, B) be spaces.

1. Let $f_i \in \hom((Y, B), (X_i, A_i))$ for every $i \in I$. Then there exists a unique morphism $h \in \hom((Y, B), \prod_{i \in I}(X_i, A_i))$ such that $f_j = h \circ \pi_j$ for every $j \in I$.
2. Let $f_i \in \hom((X_i, A_i), (Y, B))$ for every $i \in X_i$. Then there exists a unique morphism $h \in \hom(\coprod_{i \in I}(X_i, A_i), (Y, B))$ such that $f_j = i_j \circ h$ for every $j \in I$.

Also the internalization of $\mathfrak{Top}_2$ is of interest. For this we have to interpret the set $\hom((X, A), (Y, B))$ of morphisms from (X, A) to (Y, B) as a pair (Z, C) which itself is a pair of spaces. A construction similar to the one of Section 2.1.2 is possible:

Definition 7 Let $(X, A), (Y, B) \in \mathfrak{Top}_2$, and let $Z := \{f \in Y^X : f(A) \subseteq B\}$ be equipped with the inherited topology of Y^X, i.e., with the compact open topology. With $C := B^X$, regarded as a subspace of Z, define

$$(Y, B)^{(X,A)} := (Z, C).$$

Theorem 6 The functor

$$\mathrm{Hom}\colon \mathfrak{Top}_2^* \times \mathfrak{Top}_2 \to \mathfrak{Top}_2, \qquad \mathrm{Hom}((X, A), (Y, B)) := (Y, B)^{(X,A)}$$

factors through $\mathfrak{Top}_2$. This means that $|\cdot| \circ \mathrm{Hom} = \hom$.

Theorem 7 (Exponential Law in $\mathfrak{Top}_2$) Let $(X, A), (Y, B), (Z, C) \in \mathfrak{Top}_2$, X Hausdorff, Y locally compact T_2. Then

$$(Z, C)^{(X,A)\times(Y,B)} \cong \left((Z, C)^{(Y,B)}\right)^{(X,A)}$$

in $\mathfrak{Top}_2$, where

$$(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y).$$

2.3 The Category of Pointed Spaces

A *pointed space* is a pair (X, x_0) where $X \in \mathfrak{Top}$ is a space and $x_0 \in X$ a *base point*. By $\mathfrak{Top}_p$ we denote the *category of pointed spaces*. Its objects are the pointed spaces (X, x_0), and the morphisms from (X, x_0) to (Y, y_0) are the continous maps $f\colon X \to Y$ mapping x_0 to y_0.

The category $\mathfrak{Top}_p$ is a subcategory of $\mathfrak{Top}_2$ which can be seen by rephrasing (X, x_0) as $(X, \{x_0\})$. Moreover, there is the so called *contraction functor* $c\colon \mathfrak{Top}_2 \to \mathfrak{Top}_p$ which maps a pair (X, A) to the pointed space $(X/A, A)$, i.e., c contracts A to a single point.

As in $\mathfrak{Top}$ and in $\mathfrak{Top}_2$, we are interested in products, coproducts, and internalization.

For products, the situation is as expected. If the product of pointed spaces X_i $(i \in I)$ is defined by

$$\prod_{i\in I}(X_i, x_0^{(i)}) := \Big(\prod_{i\in I} X_i, (x_0^{(i)})_{i\in I}\Big)$$

then the analogon of Theorems 1 (1) and 5 (1) also holds.

However, for coproducts, the situation is more complicated. The problem is that the naive componentwise definition

$$\coprod_{i\in I}(X_i, x_0^{(i)}) \stackrel{?}{:=} \Big(\coprod_{i\in I} X_i, \{x_0^{(i)} : i \in I\}\Big)$$

does not work since $\{x_0^{(i)} : i \in I\}$ is not a singleton. The solution is to identify (glue) the base points to one single point. This leads to the concept of *wedge sums.*

Definition 8 Let $(X, x_0), (Y, y_0) \in \mathfrak{Top}_p$. Then

$$(X, x_0) \vee (Y, y_0) := \Big((X \dot\cup Y)/\{x_0, y_0\}, [x_0]\Big)$$

is called the *wedge sum* of (X, x_0) and (Y, y_0). Here $X \dot\cup Y$ means the disjoint union (coproduct), $[x_0]$ means the equivalence class of x_0, which is simply $\{x_0, y_0\}$.

More general, if $(X_i, x_0^{(i)})$ $(i \in I)$ is a family of pointed spaces, we define

$$\bigvee_{i\in I}(X_i, x_0^{(i)}) := \Big(\textstyle\coprod_{i\in I} X_i/\{x_0^{(i)} : i \in I\}, \{x_0^{(i)} : i \in I\}\Big)$$

to be the *wedge sum* of the pointed spaces $(X_i, x_0^{(i)})$ $(i \in I)$.

The internalization of $\mathfrak{Top}_p$ does not pose any problems if we regard pointed spaces $(X, x_0) \in \mathfrak{Top}_p$ as pairs $(X, \{x_0\}) \in \mathfrak{Top}_2$. The results carry over from Section 2.2.

In particular, we have also an exponential law in $\mathfrak{Top}_p$:

Theorem 8 Let $(X, x_0), (Y, y_0), (Z, z_0) \in \mathfrak{Top}_p$, Y locally compact T_2, X Hausdorff. Then

$$(Z, z_0)^{(X,x_0)\times(Y,y_0)} \cong \Big((Z, z_0)^{(Y,y_0)}\Big)^{(X,x_0)}$$

in $\mathfrak{Top}_p$ where

$$(X, x_0) \times (Y, y_0) := (X \times Y, X \vee Y).$$

3 Homotopy

3.1 Homotopic Maps

Definition 9 Given two spaces $X, Y \in \mathfrak{Top}$ and $f, g \in C(X, Y)$. f and g are called *homotopic*, written as $f \simeq g$, if there exists a continous function $H \in C(X \times [0,1], Y)$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. Such an H is called a *homotopy* to f and g.

A homotopy H has to be understood intuitively as a continous deformation of $f(X)$ into $g(X)$. The second parameter can be interpreted as the time: In the beginning, H behaves like f, but at time 1, it behaves like g.

Example 3 Given $X = Y = \mathbb{R}^2$ with the usual topology. The maps

$$f\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = x \cdot y, \qquad g\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x, y) = x + y$$

are homotopic. It is shown by providing a homotopy.

$$H\colon \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2, \quad H(x, y, t) := (1 - t) \cdot x \cdot y + t \cdot (x + y)$$

obviously does the job, and hence $f \simeq g$. In fact, a similar construction works for every choice of f and g in $C(\mathbb{R}^2, \mathbb{R}^2)$, for all such f and g we have $f \simeq g$.

Theorem 9 Homotopy $\simeq$ is a congruence relation on $C(X, Y)$ $(X, Y \in \mathfrak{Top})$, i.e., it is an equivalence relation and it is compatible with composition.

The rest of this sumary will be dedicated to the study of the equivalence classes of $\simeq$, and to the definition of homotopy not only for $\mathfrak{Top}$, but also for the categories $\mathfrak{Top}_2$ and $\mathfrak{Top}_p$.

3.2 Homotopy Categories

3.2.1 The Category of Spaces

Definition 10 The category $\mathfrak{Top}' := \mathfrak{Top}/_{\simeq}$ is called the *homotopy category*. Its objects are the spaces $X \in \mathfrak{Top}$, the morphism set $M(X, Y)$ to spaces $X, Y \in \mathfrak{Top}$ is the set $C(X, Y)/_{\simeq}$, also written as $[X, Y]$. The composition is defined by the composition of representatives: $[f] \circ [g] := [f \circ g]$.

The 0-component $[x \mapsto x_0]$, the homotopy class of a constant map, will be denoted by 0.

There is a natural functor relating $\mathfrak{Top}$ and $\mathfrak{Top}'$ to each other which maps a space $X \in \mathfrak{Top}$ to the same space $X \in \mathfrak{Top}'$, and maps $f \in C(X, Y)$ to its homotopy class $[f] \in [X, Y]$. Two spaces X, Y isomorphic in $\mathfrak{Top}'$ are called *homotopic equivalent*, written as $X \simeq Y$. Two spaces X, Y being isomorphic in $\mathfrak{Top}$ are also isomorpic in $\mathfrak{Top}'$, i.e., they are homotopic equivalent. Isomorphisms in $\mathfrak{Top}'$ are called *homotopy equivalences.*

3.2.2 Pairs and Pointed Spaces

Homotopy of morphisms in the category of pairs is defined analogously to homotopy of morphisms in $\mathfrak{Top}$.

Definition 11 Let $(X, A), (Y, B) \in \mathfrak{Top}_2$ and $f, g \in (X, A)^{(Y,B)}$. f and g are called *homotopic* if there exists a continous map

$$H : (X, A) \times I \to (Y, B)$$

with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. If f is homotopic to g, we write again $f \simeq g$.

As in $\mathfrak{Top}$, the idea is to continously deform the image $f(X)$ into the image $g(X)$. The additional requirement here is that $H(A, t) \subseteq B$ for each $t \in [0, 1]$, i.e., H is meant to be continous wrt. the topology of $(Y, B)^{(X,A)\times(I,I)}$.

It is not a surprise that also homotopy in $\mathfrak{Top}_2$ is a congruence relation. This allows us to define $\mathfrak{Top}_2' := {}^{\mathfrak{Top}_2}/_{\simeq}$ in analogy to Def. 10.

Again, the category of pointed spaces can be treated by regarding a pointed space (X, x_0) as a pair $(X, \{x_0\})$. The definition of homotopy and the category $\mathfrak{Top}_p' = {}^{\mathfrak{Top}_p}/_{\simeq}$ are according to this point of view.

If $f, g : (X, x_0) \to (Y, y_0)$ are homotopic by a homotopy

$$H : (X, x_0) \times I \to (Y, y_0),$$

then $H(x_0, t) = y_0$ for all $t \in [0, 1]$.

3.3 Applications

3.3.1 Contractibility

We start with recalling the definitions of some basic topological notions.

Definition 12 Let $X \in \mathfrak{Top}$.

1. Two points $x, x' \in X$ are 0-*connected* (or *path connected*), if there exists an $\alpha \in X^{[0,1]}$ with $\alpha(0) = x$ and $\alpha(1) = x'$.
2. Being 0-connected is an equivalence relation on X. The class $[x]$ of x is called the 0-component of x in X. Further, we write

$$\pi_0(X) := \{[x] : x \in X\} = {}^{X}/_{\sim}$$

for the set of 0-components.

3. X is 0-*connected* if there is only one 0-component (namely X itself).

The notion of *contractible* expresses the fact that a space can be continously deformed (contracted) to a single point. Formally, this is defined in the following definition.

Definition 13 Let $X, Y \in \mathfrak{Top}$.

1. $f \in C(X, Y)$ is 0-*homotopic*, if it is homotopic to a constant map.
2. X is *contractible* if the identity 1_X is 0-homotopic.

Example 4 The approach of Example 3 shows that $\mathbb{R}^n$ is contractible. More general: Every star shaped subset of $\mathbb{R}^n$ is contractible.

Similar considerations in a slightly more general setting lead to the following theorem, which sumarizes some first results.

Theorem 10 Let $X, Y \in \mathfrak{Top}$.

1. If X is contractible, then X is 0-connected.
2. If Y is contractible, then $[X, Y] = \{0\}$.
3. If X is contractible, then $[X, Y] = \pi_0(Y)$.

Example 5 If $Y \neq \emptyset$ is 0-connected, then $[\mathbb{R}^n, Y] = \{0\}$. Conversely, for any $X \in \mathfrak{Top}'$, we have $[X, \mathbb{R}^n] = \{0\}$.

More interesting examples are collected in the following section.

3.3.2 Nontrivial Examples

First we define some spaces. We already used the n dimensional real space $\mathbb{R}^n$ with the classical topology. The spaces we are mainly interested in are all subspaces of $\mathbb{R}^n$ with the inherited topology. We identify $\mathbb{C}$ with $\mathbb{R}^2$.

Definition 14

1. $I := [0, 1] \subseteq \mathbb{R}$ is called the *unit interval.*
2. $\mathbb{D}^n := \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ is the n dimensional *disc.*
3. $\mathring{\mathbb{D}}^n := \{x \in \mathbb{R}^n : ||x|| < 1\}$ is the *open* n dimensional disc.
4. $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ is the n dimensional real *sphere.*

The spaces defined in the definition above are interrelated in various ways. For instance: $\mathbb{S}^1 \cong I/\partial I$, $\mathring{\mathbb{D}}^n = (\mathbb{D}^n)^0$, $\mathbb{S}^n = \partial\mathbb{D}^{n+1}$, etc.
The points $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$ are called *northpole* and *southpole*, respectively, of $\mathbb{D}^n$ or $\mathbb{S}^n$. If $\mathbb{S}^n$ or $\mathbb{D}^n$ are regarded as pointed spaces, we will always take the northpole as the base point. The canonical base point of $\mathring{\mathbb{D}}^n$ is the origin.

Example 6 Let $X \in \mathfrak{Top}$ and $f: X \to \mathbb{S}^n$ such that there exists an $s_0 \in \mathbb{S}^n \setminus f(X)$, i.e., f is not surjective. Then f is 0-homotopic.

We now study the structure of $[\mathbb{S}^1, \mathbb{S}^1]$.

Consider a continous function $\varphi\colon I \to \mathbb{R}$ with $\varphi(0) = 0$ and $\varphi(1) = n \in \mathbb{Z}$. Then $\exp(2\pi \mathrm{i} \varphi(0)) = \exp(2\pi \mathrm{i} \varphi(1)) = 1$, which means that the diagram

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[dr] & \mathbb{R} \arrow[r, "\exp(2\pi \mathrm{i} t)"] & \mathbb{S}^1 \\ & {}^{I}\!/_{\partial I} \cong \mathbb{S}^1 \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

commutes. The function φ gives rise to a corresponding function

$$\hat{\varphi}\colon \mathbb{S}^1 \to \mathbb{S}^1, \qquad \hat{\varphi}(\exp(2\pi \mathrm{i} t)) = \exp(2\pi \mathrm{i} \varphi(t)).$$

Theorem 11 Given $f \in C(\mathbb{S}^1, \mathbb{S}^1)$ there is a unique map $\varphi \in C((I, 0), (\mathbb{R}, 0))$ with the property that $f(z) = f(1) \cdot \hat{\varphi}(z)$.

Since φ is uniquely determined for a given function f on the sphere, to each f there corresponds a number $\varphi(1)$. For reasons of continuity, this number must be an integer, which we will call the *degree* of f.

Definition 15 Let $f \in C(\mathbb{S}^1, \mathbb{S}^1)$ and let $\varphi \in C(I, \mathbb{R})$ be the uniquely determined map from Theorem 11. Then $\deg f := \varphi(1)$ is called the *degree* of f.

Theorem 12

1. $f, g \in C(\mathbb{S}^1, \mathbb{S}^1)$ are homotopic if and only if $\deg(f) = \deg(g)$.
2. For the monomial function $f\colon z \mapsto z^n$ $(n \in \mathbb{Z})$ we have $\deg(f) = n$.
3. The groups $([\mathbb{S}^1, \mathbb{S}^1], \circ)$ and $(\mathbb{Z}, \cdot)$ are isomorphic as monoids.

Sumarizing the results, we can say that every continous map f form $\mathbb{S}^1$ to itself starts at some point x on the sphere, and turns several times around it, eventually changing its direction in between, and finally comes back to the starting point x. f does exacty $\deg f$ rotations around, if clockwise rounds count -1, rounds in the other direction count 1, and every corresponding parts cancel if the direction has changed.

Isomorphisms always have degree ± 1, but conversely, a map of degree ± 1 need not be an isomorphism.

3.3.3 Consequences

Theorem 13 The sphere $\mathbb{S}^1$ is 0-connected but not contractible.

Theorem 14 There is no continous map $f\colon \mathbb{D}^2 \to \mathbb{S}^1$ with $f|_{\mathbb{S}^1} = \mathrm{id}_{\mathbb{S}^1}$.

Theorem 15 (Brower's Theorem) Every $f \in C(\mathbb{D}^n, \mathbb{D}^n)$ has a fixed point.

Theorem 16 (Fundamental Theorem of Algebra) Every nontrivial polynomial has a root in $\mathbb{C}$.

Definition 16 To $f \in C(\mathbb{S}^1, \mathbb{C})$ and $z \in \mathbb{C} \setminus f(\mathbb{S}^1)$ let

$$w_z : \mathbb{S}^1 \to \mathbb{S}^1, \qquad w_z(x) := \frac{f(x) - z}{|f(x) - z|},$$

and call $w(f, z) := \deg(w_z)$ the *winding number* of f around z.

Theorem 17 Let $f \in C(\mathbb{S}^1, \mathbb{C})$.

1. If two points $z, z' \in \mathbb{C} \setminus f(\mathbb{S}^1)$ are in the same connected component, then they have the same winding number.
2. $\mathbb{C} \setminus f(\mathbb{S}^1)$ has exactly one unbounded 0-connected component K. The winding number for all $z \in K$ is 0.

Theorem 18 (Theorem of Bursuk-Ulam) For every $f \in C(\mathbb{S}^2, \mathbb{R}^2)$ there exists an $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$.

(Counter-)intuitively, the theorem of Bursuk-Ulam states that, for instance, there is always a place x on the earth ($= \mathbb{S}^2$) such that on the opposite side $-x$ there is the same pressure and temperature (taking as f the map which maps each location x to the pair of pressure and temperature).

4 Functorial Constructions

4.1 Wedge Sum and Smash Product

We consider pointed spaces $(X_i, x_0^{(i)})$ $(i \in I)$. The *wedge sum* of the spaces X_i has already been defined to

$$\bigvee_{i \in I} (X_i, x_0^{(i)}) := \left(\coprod_{i \in I} X_i / \{x_0^{(i)} : i \in I\}, \{x_0^{(i)} : i \in I\} \right)$$

It is constructed to behave like the coproduct in the sense of Theorem 1. For the intuition, the wedge sum of spaces $(X_i, x_0^{(i)})$ is the disjoint union of the X_i where all the base points $x_0^{(i)}$ are glued together.
Applying the construction to only two spaces looks as follows.

$$(X_1, x_1) \vee (X_2, x_2) = \left(X_1 \dot{\cup} X_2 / \{x_1, x_2\}, [x_1] \right)$$

Given four spaces $X_1, X_2, Y_1, Y_2 \in \mathfrak{Top}_p$ and maps $f_1 \in Y_1^{X_1}$, $f_2 \in Y_2^{X_2}$, there exists exactly one map

$$f_1 \vee f_2 \colon X_1 \vee X_2 \to Y_1 \vee Y_2$$

such that $i \circ f_1 = (f_1 \vee f_2) \circ i = i \circ f_2$ where the i denote the canonical injections.

This observation allows to regard the wedge sum as a functor

$$\vee\colon \mathfrak{Top}_p \times \mathfrak{Top}_p \to \mathfrak{Top}_p.$$

It maps a pair (X_1, X_2) of pointed spaces to its wedge sum $X_1 \vee X_2$, and a pair $(f_1, f_2) \in X_1^{Y_1} \times X_2^{Y_2}$ of morphisms to $f_1 \vee f_2$.

Does this also work for the product? Consider pointed spaces $(X_i, x_0^{(i)})$ $(i \in I)$ and for each $j \in I$ morphisms

$$\beta_j\colon (X_j, x_0^{(j)}) \to \prod_{i\in I} X_i, \quad \beta_j(u) := (u_i)_{i\in I}$$

where $u_i = x_0^i$ $(i \neq j)$ and $u_i = u$ $(i = j)$. The maps β_j can be regarded as the maps which map a space X_j to the corresponding "coordinate axis" in $\prod_{i\in I} X_i$. The union of the coordinate axis $\beta_j(X_j)$ is just the wedge sum of the X_j. Clearly $\beta_j(X_j) \cong X_j$, and the intersection of $\beta_i(X_i)$ and $\beta_j(X_j)$ $(i \neq j)$ contains only the origin $(x_0^{(i)})_{i\in I}$ which is by construction the base point of the wedge sum.

The *smash product* is obtained from the product space by identifying its coordinate axes.

Definition 17 Let $(X_i, x_0^{(i)})$ $(i \in I)$ be a family of pointed spaces. Then

$$\bigwedge_{i\in i}(X_i, x_0^{(i)}) := \prod\nolimits_{i\in I}(X_i, x_0^{(i)}) / \bigvee\nolimits_{i\in I}(X_i, x_0^{(i)})$$

is called the *smash product* of the $(X_i, x_0^{(i)})$.

Now take again four pointed spaces $X_1, X_2, Y_1, Y_2 \in \mathfrak{Top}_p$ with base points x_1, x_2, y_1, y_2, respectively, and two morphisms $f_1 \in Y_1^{X_1}$, $f_2 \in Y_2^{X_2}$. The *product map*

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$$

is simply defined component wise by $(f_1 \times f_2)(a, b) := (f_1(a), f_2(b))$.

In particular, since f_1 and f_2 are morphisms of $\mathfrak{Top}_p$, their product $f_1 \times f_2$ maps axis points to axis points, e.g., $(f_1 \times f_2)(x_1, t) = (y_1, f_2(t))$. This allows to define the *smash product* of f_1 and f_2 to be the morphism

$$f_1 \wedge f_2\colon X_1 \wedge X_2 \to Y_1 \wedge Y_2$$

which maps $[a] \times [b]$ to $[f_1(a)] \times [f_2(b)]$.

This makes the smash product a functor

$$\wedge\colon \mathfrak{Top}_p \times \mathfrak{Top}_p \to \mathfrak{Top}_p$$

which — similar to the situation with the wedge sum — maps a pair (X_1, X_2) of pointed spaces to the pointed space $X_1 \wedge X_2$, and a pair (f_1, f_2) of morphisms to $f_1 \wedge f_2$.

Theorem 19 Let $X, Y \in \mathfrak{Top}_p$, X Hausdorff, Y Hausdorff and locally compact. Then

$$[X \wedge Y, Z] \cong [X, Z^Y].$$

4.2 Suspension and Loop space

Definition 18 If $X = (X, x_0) \in \mathfrak{Top}_p$ is a pointed space, then

$$SX := S(X, x_0) := \mathbb{S}^1 \wedge X = (\mathbb{S}^1, 1) \wedge (X, x_0)$$

is called the *suspension* of X.

Example 7 When trying to imagine concrete examples for the suspension, one soon runs out of dimensions. (Try to draw $S\mathbb{D}^2$, for instance.) Intuition is hence restricted to very trivial spaces only. We consider $X = (I, 0)$. $\mathbb{S}^1 \times I$ is the hollow cylinder without covers. $\mathbb{S}^1 \vee I$ is obtained by glueing I and $\mathbb{S}^1$ together at their basepoints. In order to get to $\mathbb{S}^1 \wedge I$, we have to take the cylinder $\mathbb{S}^1 \times I$ and to glue together the contained sphere $\mathbb{S}^1 \times 0$ and the edge $1 \times I$. The result is a hollow hemisphere with its base point at the equator.

For another example, take $X = (x_0, x_0)$ to be the pointed space only consisting of one point. Then $SX \cong X$, because $\mathbb{S}^1 \times X \cong \mathbb{S}^1$ and $\mathbb{S}^1 \vee X \cong \mathbb{S}^1$ as well.

Example 8 $\mathbb{S}^1 \wedge \mathbb{S}^n \cong \mathbb{S}^{n+1}$. This implies $S\mathbb{S}^n \cong \mathbb{S}^{n+1}$ and $S^n X = \mathbb{S}^n \wedge X$.

The suspension admits also a functor $S\colon \mathfrak{Top}_p \to \mathfrak{Top}_p$ which acts on spaces as defined above, and which maps a morphism $f\colon X \to Y$ to a morphism $Sf\colon SX \to SY$ which acts on x by

$$Sf([z, x]) = (1 \wedge f)([z, x]) = [(1 \times f)(z, x)] = [z, f(x)].$$

The suspension functor can hence be viewed as the composition of the embeding functor $\mathfrak{Top}_p \to \mathfrak{Top}_p \times \mathfrak{Top}_p$ mapping X to $(\mathbb{S}^1, X)$ and the smash product functor $\mathfrak{Top}_p \times \mathfrak{Top}_p \to \mathfrak{Top}_p$, $(X, Y) \mapsto X \wedge Y$.

Definition 19 To $X \in \mathfrak{Top}_p$, the space $\Omega X := X^{\mathbb{S}^1}$ is called the *loop space* of X.

Example 9 If X consists of only one point, then $\Omega X \cong X$. The loop space $\Omega\mathbb{S}^1$ was studied in Section 3.3.2.

The loop spaces of X consists of all closed paths from x_0 to itself. Ω can be regarded as a functor from $\mathfrak{Top}_p \to \mathfrak{Top}_p$, called the *loop functor.* Indeed, it is

$$\Omega\colon \mathfrak{Top}_p \to \mathfrak{Top}_p, \quad \Omega X = \mathrm{Hom}(\mathbb{S}^1, X)$$

where $\mathrm{Hom}\colon \mathfrak{Top}_p^* \times \mathfrak{Top}_p$ is the covariant hom functor in $\mathfrak{Top}_p$, cf. Section 2.3.

The following theorem interrelates the functors S and Ω. It is a direct consequence of Theorem 19.

Theorem 20 Let $X, Y \in \mathfrak{Top}_p$, X Hausdorff. Then

$$[SX, Y] \cong [X, \Omega Y].$$

5 Homotopy Groups

5.1 Groups and H-Groups

Similar to $\mathfrak{Top}_p$, we can define the category $\mathfrak{Set}_p$ of *pointed sets*, whose objects are pairs (S, s) where S is a set and $s \in S$, and whose morphisms are the maps mapping base points to base points. In this sense, let $(G, e) \in \mathfrak{Set}_p$ be a pointed set, and let

$$\mu\colon G \times G \to G, \qquad \nu\colon G \to G$$

be morphisms in $\mathfrak{Set}_p$. We call (G, e, μ, ν) a *group*, if the following diagrams are commutative. In these diagrams, we mean e to be the constant map mapping everything to the base point e, and id is the identity.

$$\begin{tikzcd} G \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "\mathrm{id}"] \\ & G & \end{tikzcd} \tag{A}$$

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd} \tag{B}$$

$$\begin{tikzcd} G \arrow[r, "\nu \times \mathrm{id}"] \arrow[dr, "e"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "\mathrm{id} \times \nu"'] \arrow[dl, "e"] \\ & G & \end{tikzcd} \tag{C}$$

These diagrams express the behaviour of the identity (A), the associativity (B), and the behaviour of inverses (C). If, furthermore, the diagram

$$\begin{tikzcd} G \times G \arrow[rr, "T"] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\ & G & \end{tikzcd} \tag{D}$$

commutes, where T is meant to be the map which interchanges dimensions, i.e., (D) expresses commutativity, then G is naturally called *abelian group*.

There is nothing special to the category $\mathfrak{Set}_p$ in this definition, and the purpose of this chapter is to study groups defined by the diagrams (A), (B), and (C), but living in different categories.

Definition 20 A tuple (X, x_0, μ, ν) where $(X, x_0) \in \mathfrak{Top}_p$ and

$$\mu \colon X \times X \to X, \qquad \nu \colon X \to X$$

are morphisms preserving base points is called an *H-group*, if the diagrams (A), (B), and (C) with X in place of G and x_0 in place of e commute in $\mathfrak{Top}'_p$. If the diagrams commute in $\mathfrak{Top}_p$, then (X, x_0, μ, ν) is called a *topological* group.

Example 10 For any $Y \in \mathfrak{Top}_p$, ΩY is an H-group, if multiplication μ is defined as concatenation of loops, and inversion ν is reversion of a loop's direction.

Theorem 21 Let (K, k_0) be an H-group with multiplication $\mu \colon K \times K \to K$ and inverse $\nu \colon K \to K$, and let $(X, x_0) \in \mathfrak{Top}'_p$. If to any $f, g \in [(X, x_0), (K, k_0)]$, the product $[f] \cdot [g]$ is defined by

$$[f] \cdot [g] := [\mu \circ (f \times g) \circ \Delta],$$

with

$$\Delta \colon X \to X \times X, \qquad \Delta(x) := (x, x)$$

and if $[f]^{-1} := [\nu \circ f]$, then $([(X, x_0), (K, k_0)], \cdot, \circ^{-1})$ is a group. Its identity is $[x \mapsto k_0]$.

The association $(X, x_0) \mapsto [(X, x_0), (K, k_0)]$ defines a contravariant functor from $\mathfrak{Top}^*_p$ to $\mathfrak{Group}$.

5.2 $\mathfrak{A}$-Groups

Rather than doing the definition of groups for every pointed category separately, it is preferable to define the notion of group directly in an arbitrary setting leaving the category unspecified. But for general categories, the definition does not work. We need *pointed categories.*

Definition 21 A category $\mathfrak{A}$ is *pointed* if there is a *choice function* which selects from each morphism set $M(X, Y)$ a morphism m such that the composition of m with any other morphism $m' \in M(Y, Z)$ or $m'' \in M(Z, X)$ gives the selected morphism of $M(X, Z)$ or $M(Z, Y)$, respectively.

If a category has a choice function, then this choice function is uniquely determined. If a pointed category has a zero object 0, i.e., an object where $|M(0, X)| = |M(X, 0)| = 1$ for any X, then the selected morphism $m \in M(X, Y)$ is the morphism obtained by composition of the element of $M(X, 0)$ with the element of $M(0, Y)$. This morphism will be denoted by 0_{XY} or simply by 0.

For instance, the categories $\mathfrak{Set}_p$ and $\mathfrak{Top}_p$ are pointed, their zero objects are the sets/spaces $(\{x_0\}, x_0)$, and the selected morphism $0_{(X,x_0),(Y,y_0)}$ maps every $x \in X$ to $y_0 \in Y$. The categories $\mathfrak{Set}$ and $\mathfrak{Top}$ are not pointed.

We will now define $\mathfrak{A}$-groups to be groups in the pointed category $\mathfrak{A}$.

Definition 22 Let $\mathfrak{A}$ be a pointed category whose object class is closed under taking binary products. A triple (G, μ, ν), where $G \in \mathfrak{A}$, $\mu\colon G \times G \to G$, $\nu\colon G \to G$ are morphisms, is called an $\mathfrak{A}$-*group* if the diagrams (A), (B), and (C) commute, where e is meant to be the selected morphism from G to itself. If, in addition, (D) is commutative, then G is called *abelian.*

If G and H are $\mathfrak{A}$-groups, and $\varphi\colon G \to H$ is a morphism that makes the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\varphi \times \varphi"] \arrow[d, "\mu_G"'] & H \times H \arrow[d, "\mu_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

commute, then φ is called $\mathfrak{A}$-homomorphism between G and H.

Example 11 An H-group is a $\mathfrak{Top}'_p$-group, because $\mathfrak{Top}'_p$ is a pointed category with binary products.

Theorem 22 If $\mathfrak{A}$ is a pointed category and $K \in \mathfrak{A}$ is an $\mathfrak{A}$-group, then the contravariant hom functor factors through $\mathfrak{Group}$, i.e., the diagram

$$\begin{tikzcd} \mathfrak{A}^* \times \mathfrak{A} \arrow[r, "\hom"] & \mathfrak{Set} \\ \mathfrak{A}^* \arrow[u, "X \mapsto (K,X)"] \arrow[ur, "\hom(\cdot, K)"] \arrow[r, "\cong"'] & \mathfrak{Group} \arrow[u, "|\cdot|"'] \end{tikzcd}$$

commutes.

Definition 23 If $\mathfrak{A}$ is a pointed category with binary coproducts, then $\mathfrak{A}^*$-groups are called $\mathfrak{A}$-*cogroups.*

If (K, k_0, μ, ν) is an $\mathfrak{A}$-cogroup, then they fulfill the diagrams obtained from (A), (B), and (C) by reversing the arrow directions, replacing K by G, e by k_0, and $\times$ by the binary coproduct $\sqcup$.

Note the reverse of the direction: A multiplication μ takes values in K and has images in $K \sqcup K$.

Example 12 $\boxed{\text{TO BE: cogroup example}}$ (Is there any nontrivial cogroup?)

The notion of cogroups allows to state an analogon to Example 10:

Theorem 23 For any pointed space $X \in \mathfrak{Top}_p$, the suspension SX is a $\mathfrak{Top}'_p$-cogroup. $\boxed{\text{TO BE: what is } \mu \text{ and } \nu \text{ here?}}$ (Do the diagrams commute here?)

There is an analogon to Theorem 21 for cogroups. By H-cogroup, we mean $\mathfrak{Top}'_p$-cogroup.

Theorem 24 Let (K, k_0) be an H-cogroup with comultiplication $\mu\colon K \to K \wedge K$ and inverse ν. The for every $(Y, y_0) \in \mathfrak{Top}_p$, the set $[K, k_0, Y, y_0]$ becomes a group by the definition

$$[f] \cdot [g] := [\nabla \circ (f \wedge g) \circ \mu]$$

where ∇ maps $Y \wedge Y$ canonically to Y. The identity of that group is $[k \mapsto y_0]$, the inverse of $[f]$ is $[f]^{-1} = [f \circ \nu]$.
The association $(X, x_0) \mapsto [(K, k_0), (X, x_0)]$ defines a covariant functor from $\mathfrak{Top}_p$ to groups.

The interesting result is that Theorem 20 also holds for groups:

Theorem 25 Let $X, Y \in \mathfrak{Top}_p$, X Hausdorff. Then $[SX, Y], [X, \Omega Y]$ are groups, and

$$[SX, Y] \cong [X, \Omega Y]$$

in $\mathfrak{Group}$.

Theorem 26 Let $X, Y \in \mathfrak{Top}_p$. Then

1. $[SX, \Omega Y]$ is abelian.
2. $\Omega^n Y$ is a commutative H-group for $n \geq 2$.
3. $S^n X$ is a commutative H-cogroup for $n \geq 2$.

6 The Theorem of Seifert-van Kampen

6.1 Free Products

Let $X \in \mathfrak{Set}$ and X' be a disjoint copy of X. Consider the sequences (words) of the elements of $X \cup X'$. Such a word is called *reduced* if it does not contain xx' or $x'x$ for any $x \in X$ as a subword. Let's declare that $x'' = x$ for convenience. Let F_X be the set of all these reduced words over the alphabet $X \cup X'$.
The operation $\circ$ on F_X is defined as concatenation plus reduction. This means that if (a, b, c) and (d, e, f) are reduced words in F_X, then $(a, b, c) \circ (d, e, f)$ is (a, b, c, d, e, f) or $(a, b) \circ (e, f)$ if $c = d'$. $(F_X, \circ)$ is a group with identity $()$ and with $(x_1, \ldots, x_n)^{-1} = (x'_n, \ldots, x'_1)$.

Example 13 A group G can be specified by a set X of *generators* and a set $K \subseteq F_X$ of *relations* in the way $G := {}^{F_X}\!/_{K}$. For instance for $X = \{a, b\}$ and $K = \{aba'b'\}$ we have $\mathbb{Z} \times \mathbb{Z} \cong {}^{F_X}\!/_{K}$.

Let G_1, G_2 be groups. Without loss of generality, we may assume that G_1 and G_2 only share the unit element, which we will denote by 1.
The *free product* $G_1 * G_2$ of G_1 with G_2 the set of all finite sequences $(x_1, \ldots, x_n)$ over $G_1 \cup G_2$ where consecutive elements x_i, x_{i+1} do not belong to the same generating group G_1 or G_2. None of the x_i appearing in a sequence be the unit element 1. For $a, b \in G_1 * G_2$, $a \circ b$ is defined as concatenation where $\ldots, a_n, b_1, \ldots$ is replaced by $\ldots, a_n \cdot b_1, \ldots$ if a_n and b_1 belong to the same underlying group.

Definition 24 Let G_1, G_2 be groups with $G_1 \cap G_2 = \{1\}$.

1. If $X = G_1 \cup G_2$ and
$$K := \{(a, b, (ab)^{-1}) : a, b \in G_1\} \cup \{(a, b, (ab)^{-1}) : a, b \in G_2\},$$
then $G_1 * G_2 := {}^{F_X}\!/_{K}$ is called the *free product* of G_1 and G_2.

2. If H is another group and $f_1 \colon G_1 \to H$, $f_2 \colon G_1 \to H$ are morphisms, then $f_1 * f_2 \colon G_1 * G_2 \to H$ is defined to be the morphism which maps a sequence $(x_1, \ldots, x_n) \in G_1 * G_2$ to $\prod_{i=1}^{n} f(x_i) \in H$
$$f(x) := \begin{cases} f_1(x) & \text{if } x \in G_1 \\ f_2(x) & \text{if } x \in G_2 \end{cases}.$$

Example 14 $\mathbb{Z}_2 * \mathbb{Z}_2$ contains the elements
$$(), (a), (b), (ab), (ba), (aba), (bab), (abab), (baba), \ldots$$
with which computation can be done as expected. For instance
$$aba \cdot abab = abaabab = abbab = aab = b.$$
$\mathbb{Z}_2 * \mathbb{Z}_2$ is called the *infinite Dieder group.*

6.2 Seifert-van Kampen's Theorem

Definition 25 By $\pi_1(X, x_0) := \Omega X$ we denote the *homotopy group* of $(X, x_0) \in \mathfrak{Top}_p$.

Definition 26 A space $X \in \mathfrak{Top}$ is *simply connected* if it is path connected and every closed path is zero homotopic.

Let (X, x_0) be a pointed space, $U, V \subseteq X$ open, and $U \cap V \supseteq \{x_0\}$ path connected. Let $i \colon U \cap V \hookrightarrow U$, $i' \colon U \hookrightarrow X$, $j \colon U \cap V \hookrightarrow V$, $j' \colon V \hookrightarrow X$ denote the injection maps from $U \cap V$ to U and V and from there to X. These maps correspond to homomorphisms between the corresponding homotopy groups. We will denote them by i_*, i'_*, j_*, j'_*, respectively.

$$\begin{tikzcd} U \arrow[r, "i"] & X \\ U \cap V \arrow[u, "i'"] \arrow[r, "j'"'] & V \arrow[u, "j"'] \end{tikzcd} \qquad \begin{tikzcd} \pi_1(U, x_0) \arrow[r, "i_*"] & \pi_1(X, x_0) \\ \pi_1(U \cap V, x_0) \arrow[u, "i'_*"] \arrow[r, "j'_*"'] & \pi_1(V, x_0) \arrow[u, "j_*"'] \end{tikzcd}$$

Theorem 27 (Seifert-van Kampen) Let $(X, x_0), U, V, i, i', i_*, i'_*, j, j', j_*, j'_*$ be as above. Then

$$i_* * j_* \colon \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

is surjective,

$$\pi(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) / \ker(i_* * j_*),$$

and

$$\ker(i_* * j_*) = \mathcal{N}(\{i'_*(\omega) j'_*(\omega) : \omega \in \pi_1(U \cap V, x_0\}),$$

where $\mathcal{N}(S)$ denotes the normal subgroup generated by S.