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Algorithmic Algebraic Geometry
Solution 5

Let $F \in k[x_1, \ldots, x_n]$ be a homogenous polynomial of degree d. We have to show EULER's equation

$$d \cdot F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F.$$

Proof. It suffices to prove the claim for one single monomial, since multiplication by the constant d is distributive and the sum on the righthand side is commutative and associative, i.e. can be split into several sums, one per monomial.

So, let

$$m = \prod_{i=1}^{n} x_i^{e_i}$$

be a monomial of F. Since F is homogenous, we know that

$$\sum_{i=1}^{n} e_i = \deg m = \deg F = d. \tag{1}$$

Furthermore, we have

$$x_j \frac{\partial}{\partial x_j} m = x_j \frac{\partial}{\partial x_j} \prod_{i=1}^{n} x_i^{e_i} = x_j \cdot e_j \cdot x_j^{e_j - 1} \prod_{\substack{i=0 \\ i \neq j}}^{n} x_i^{e_i} = e_j \prod_{i=1}^{n} x_i^{e_i} = e_j \cdot m \tag{2}$$

and hence

$$\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} m \overset{(2)}{=} \sum_{j=1}^{n} (e_j \cdot m) = m \cdot \sum_{j=1}^{n} e_j \overset{(1)}{=} m \cdot d.$$

This completes the proof. ∎