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Algorithmic Algebraic Geometry
Solution 3

a) Let $R = \mathbb{Q}[x, y, z]$ and $F = x^2 + y^2 + z^2 + 3 - 10xyz \in R$. In order to find all lines on the surface $F = 0$, we write

$$F(at + b, ct + d, t) = A(a, b, c, d)t^3 + B(a, b, c, d)t^2 + C(a, b, c, d)t + D(a, b, c, d)$$

and hunt for (a, b, c, d) with $A = B = C = D = 0$. Using Singular, we can compute the primary decomposition of the ideal $\langle A, B, C, D\rangle \trianglelefteq \mathbb{Q}[a, b, c, d]$ which gives the following set of prime ideals as result:

$$\begin{array}{cc}
\langle d, c-3, b-1, a\rangle, & \langle 5d-3, c, 5b-4, a-1\rangle, \\
\langle d+1, c, b, a+3\rangle, & \langle 5d-4, c-1, 5b-3, a\rangle, \\
\langle 5d+3, c, 5b+4, a+1\rangle, & \langle d-1, c, b, a-3\rangle, \\
\langle 5d+4, c-1, 5b-3, a\rangle, & \langle d, c+3, b+1, a\rangle, \\
\langle d, 3c-1, b-1, a\rangle, & \langle d, 3c+1, b+1, a\rangle, \\
\langle 5d-3, c, 5b+4, a-1\rangle, & \langle d+1, c, b, 3a+1\rangle, \\
\langle 5d-4, c+1, 5b+3, a\rangle, & \langle 5d+3, c, 5b-4, a+1\rangle, \\
\langle d-1, c, b, 3a-1\rangle, & \langle 5d+4, c+1, 5b+3, a\rangle.
\end{array}$$

A first bunch of solutions is given by the above set of primary ideals, namely the lines

$$l\colon \begin{pmatrix} b \\ d \\ 0 \end{pmatrix} + t \begin{pmatrix} a \\ c \\ 1 \end{pmatrix}$$

for the a, b, c, d given by one of the ideals. These are 16 lines.

Now, since F is symmetrix in x, y, z, i.e.

$$F(x, y, z) = F(x, z, y) = F(y, x, z) = F(y, z, x) = F(z, x, y) = F(z, y, x),$$

we obtain new solutions by coordinate permutation of the ones we already have. This gives $6 \cdot 16 = 96$ solutions.

Some of these 96 “solutions” discribe the same line. Given two lines $g\colon u_g + tv_g$, $h\colon u_h + tv_h$ they are equal if and only if

$$\operatorname{Rank}(v_g, v_h, u_g - u_h) = 1$$

as known from linear algebra. After deleting duplicates, 24 lines are remaining:

$$\begin{array}{ccc}
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, &
\begin{pmatrix} 4/5 \\ 3/5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, &
\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} 3/5 \\ 4/5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, &
\begin{pmatrix} -4/5 \\ -3/5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, &
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} 3/5 \\ -4/5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, &
\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, &
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix},
\end{array}$$

$$
\begin{pmatrix}-1\\0\\0\end{pmatrix}+t\begin{pmatrix}0\\-1/31\\,\end{pmatrix} \qquad \begin{pmatrix}-4/5\\3/5\\0\end{pmatrix}+t\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \begin{pmatrix}0\\-1\\0\end{pmatrix}+t\begin{pmatrix}-1/3\\0\\1\end{pmatrix},
$$

$$
\begin{pmatrix}-3/5\\4/5\\0\end{pmatrix}+t\begin{pmatrix}0\\-1\\1\end{pmatrix}, \qquad \begin{pmatrix}4/5\\-3/5\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}, \qquad \begin{pmatrix}0\\1\\0\end{pmatrix}+t\begin{pmatrix}1/3\\0\\1\end{pmatrix},
$$

$$
\begin{pmatrix}-3/5\\-4/5\\0\end{pmatrix}+t\begin{pmatrix}0\\-1\\1\end{pmatrix}, \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}3\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}1\\3\\0\end{pmatrix},
$$

$$
\begin{pmatrix}4/5\\0\\3/5\end{pmatrix}+t\begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\4/5\\3/5\end{pmatrix}+t\begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\0\\-1\end{pmatrix}+t\begin{pmatrix}-3\\1\\0\end{pmatrix},
$$

$$
\begin{pmatrix}0\\0\\-1\end{pmatrix}+t\begin{pmatrix}1\\-3\\0\end{pmatrix}, \qquad \begin{pmatrix}-4/5\\0\\-3/5\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}4/5\\0\\-3/5\end{pmatrix}+t\begin{pmatrix}-1\\1\\0\end{pmatrix}.
$$

The number of solutions should be 27, so where are the three missing ones? Consider the homogenisation

$$
F_h(x,y,z,w) := w^3 F(x/w, y/w, z/w) = x^2 w + y^2 w + z^2 w + 3w^3 - 10xyz.
$$

The points $(x,y,z) \in \mathbb{Q}^3$ with $F_h(x,y,z,0) = 0$ build "solutions at infinity." But $F_h(x,y,z,0) = -10xyz$ has obviously exactly the three solutions

$$
t\begin{pmatrix}1\\0\\0\end{pmatrix}+s\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad t\begin{pmatrix}1\\0\\0\end{pmatrix}+s\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad t\begin{pmatrix}0\\1\\0\end{pmatrix}+s\begin{pmatrix}0\\0\\1\end{pmatrix}
$$

in $\mathbb{Q}^3$, which correspond to lines at infinity in the projective space.

It follows that we have 24 real solutions and three more solutions at infinity, giving 27 solutions altogether as desired.