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Algorithmic Algebraic Geometry
Solution 1

a) Let ω be a primitive n-th root of unity, if n is odd and a $2n$-th root of unity, if n is even. Then, I claim that

$$(x_1, x_2, ..., x_n) := \begin{cases} (\omega^1, \omega^2, ..., \omega^n) & n \text{ odd} \\ (\omega^1, \omega^3, ..., \omega^{n-1}) & n \text{ even} \end{cases}$$

solves the system. The first and the last equation can be verified immediately using well-known facts about roots of unity:

- n odd:

$$\sum_{i=1}^{n} \omega^i = \frac{\omega^{n+1} - \omega}{\omega - 1} = \frac{\omega - \omega}{\omega - 1} = 0.$$
$$\prod_{i=1}^{n} \omega^i = \omega^{\sum_{i=1}^{n} i} = \omega^{\frac{n(n-1)}{2}} = \omega^0 = 1.$$

- n even:

$$\sum_{i=1}^{n} \omega^{2i-1} = \omega^{-1} \sum_{i=1}^{n} \omega^{2i} = \omega^{-1} \frac{\omega^{2(n+1)} - \omega^2}{\omega^2 - 1} = \omega \frac{\overbrace{\omega^{2n}}^{=1} - 1}{\omega^2 - 1} = 0.$$
$$\prod_{i=1}^{n} \omega^{2i-1} = \omega^{-n} \prod_{i=1}^{n} \omega^{2i} = -\prod_{i=1}^{n} (\omega^2)^i = -\omega^{n(n-1)} = -1.$$

In a similar (but lengthier) fashion, I was able to prove also the second equation,

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega^{i+j} = 0,$$

but unfortunately, I did not succeed to prove the validity of the other equations in general. `:-(`

However, if we accept my claim then every permutation of the ω^i is another solution (since $\omega^i \neq \omega^j$ for $i \neq j$) so that we have $n!$ solution altogether, as desired.