Motivation

- **Kauers** can only prove certain identities with *arbitrary many variables,* e.g.,

$$\sum_{k=1}^{n}\Big(x_k + (x_{k+1}-1)\sum_{i=1}^{k} x_i\Big)\prod_{i=1}^{k} x_i \quad = \quad x_{n+1}\Big(\sum_{k=1}^{n} x_k\Big)\prod_{k=1}^{n} x_k$$

- **Schneider** can *find* certain identities with sums and products, e.g.,

$$\sum_{k=1}^{n}\Big(\frac{1}{k} + (\frac{1}{k+1}-1)\sum_{i=1}^{k} \frac{1}{i}\Big)\prod_{i=1}^{k} \frac{1}{i} \quad \longrightarrow \quad \frac{1}{n+1}\Big(\sum_{k=1}^{n} \frac{1}{k}\Big)\prod_{k=1}^{n} \frac{1}{k}$$

- **Both together** can *find* certain identities with *arbitrary many variables,* e.g.,

$$\sum_{k=1}^{n}\Big(x_k + (x_{k+1}-1)\sum_{i=1}^{k} x_i\Big)\prod_{i=1}^{k} x_i \quad \longrightarrow \quad x_{n+1}\Big(\sum_{k=1}^{n} x_k\Big)\prod_{k=1}^{n} x_k$$

  How?

Basic Notions

- Difference ring: R, σ
- Difference field: F, σ
- Constant ring/field: $\mathbb{Q}$ (for convenience)
- Polynomial difference ring: $R = F\{x\} = F[x_0, x_1, x_2, \ldots]$, $\sigma(x_i) := x_{i+1}$.
- Free difference field: $\mathbb{F}\langle x\rangle = \mathrm{quot}(\mathbb{F}\{x\})$
- Ideals $\langle S\rangle \trianglelefteq R$, difference ideal: $\langle\!\langle S\rangle\!\rangle \trianglelefteq R$
- Difference field extension: $\mathrm{quot}(F\{t\}/\langle\!\langle S\rangle\!\rangle)$ where $\langle\!\langle S\rangle\!\rangle$ is "perfect."
- $\Pi\Sigma$-Extension: $F\{t\}/\langle\!\langle t_1 - at_0 - b\rangle\!\rangle$ with $a = 1$ or $b = 0$.
- $\Pi\Sigma$-Field: Tower of $\Pi\Sigma$-extensions

The Schneider-Equation

- **Goal:** Given an expression of nested indefinte sum and prodcuts, find a “simpler” representation
- **Karr’s algorithm** *(basic idea)*
  1. Represent the expression by an appropriate tower of $\Pi\Sigma$-extensions.
  2. Solve the *Schneider-Equation* in this field
  3. translate the solution back into $\Pi\Sigma$-expressions
- **Schneider-Equation** *(most general form)*
  - Given: $a_0, \ldots, a_m \in F$, $f_1, \ldots, f_n \in F$, (F a difference field)
  - Find: $g \in F$ and $c_1, \ldots, c_n \in F$ such that

$$a_m\sigma^m(g) + a_{m-1}\sigma^{m-1}(g) + \cdots + a_1\sigma(g) + a_0 g = c_1 f_1 + \cdots + c_n f_n \tag{S}$$

  Indeed: Find a basis for the vector space of all solutions.
- Sigma can solve the Schneider equation when F is a $\Pi\Sigma$-field.

Key observation

- The algorithms in Sigma can applied to *any* difference field $F(t)$ where
  - F is a difference field where we know how to solve (S) and
  - $F(t)$ is a $\Pi\Sigma$-extension of F.
- **Consequence:** If we can solve (S) in the free difference field $\mathbb{Q}\langle x\rangle$, then can build up a $\Pi\Sigma$ tower on this field and do summation with arbitrary many variables.
- **Todo:** Solve (S) in $\mathbb{Q}\langle x\rangle$.

Global strategy

- Similar to the classical algorithms:

$$
\begin{array}{c}
\mathbb{Q}\langle x\rangle \\
\downarrow \quad \uparrow \\
\mathbb{Q}\{x\} \\
\downarrow \quad \uparrow \\
\mathbb{Q}[x_0,\ldots,x_r] \\
\downarrow \quad \uparrow \\
\mathbb{Q}[x_0,\ldots,x_r]_{\leq d} \\
\downarrow \quad \uparrow \\
\mathbb{Q}^n
\end{array}
$$

Crash-Course in Denominator Bounding: $\mathbb{Q}\langle x\rangle \to \mathbb{Q}\{x\}$

- **Goal:** Given the a_i, f_j from the Schneider-Equation, find a polynomial d such that g is a rational solution of (S) iff $g' = dg$ is a polynomial solution of the modified Schneider-Equation

$$\frac{a_0}{d}g' + \frac{a_1}{\sigma(d)}\sigma(g') + \cdots + \frac{a_m}{\sigma^m(d)}\sigma^m(g') = c_1 f_1 + \cdots + c_n f_n \tag{S$'$}$$

- **Def.:** Given a difference field $F(t)$ and $f, g \in F[t]$, define

$$\operatorname{spread}(f,g) := \{\, m \in \mathbb{N}_0 : \gcd(f, \sigma^m(g)) \notin F \,\}$$

- **Observe:** $\operatorname{spread}(f,g)$ is finite for $f, g \in \mathbb{Q}\{x\}$ and can be easily computed.
- Define: $\operatorname{spread}(\sigma^m(a_m), a_0) = \{m_1 > m_2 > \cdots > m_s\}$ and

$$\begin{aligned}
p_1 &= \sigma^m(a_m), & p_{i+1} &= p_i/d_i,\\
q_1 &= a_0, & q_{i+1} &= q_i/\sigma^{-m_i}(d_i),\\
u_1 &= 1, & u_{i+1} &= u_i \prod_{j=0}^{m_i} \sigma^{-j}(d_i),
\end{aligned}$$

  where $d_i = \gcd(p_i, \sigma_{m_i}(q_i))$.
- **Theorem:** u_s is a denominator bound.

Order bounding: $\mathbb{Q}\{x\} \to \mathbb{Q}[x_0, \ldots, x_r]$

- **Theorem:** If ρ is the maximum order appearing in one of the a_i or f_j, then any polynomial solution g of (S) must be free of x_k for all $k > \rho - m + 1$.
- Hence it is fine to chose $r = \rho - m + 1$.

Conclusion and Open Problems

- There are several kinds of problems in Symbolic Summation:
  - $\sum_{k=1}^{n} f(k)$ (indefinite; classic)
  - $\sum_{k=1}^{n} f(n, k)$ (definite; also classic)
  - $\sum_{k=1}^{n} f(k, x_k)$ (presented today)
  - $\sum_{k=1}^{n} f(n, k, x_k)$ (“semidefinite,” but probably also possible)
  - $\sum_{k=1}^{n} f(n, k, x_k, x_n)$ (“full definite,” probably significantly more complicated)