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Algorithmic Combinatorics

Homework 9

Exercise 1 Apply the Cauchy-Frobenius Lemma in order to determine

$$|S_n \mathbin{\backslash\backslash} \{0,1\}^{\binom{[n]}{2}}|$$

for $n = 3$. (You should obtain the result 4.)

Write $G = S_3$ and $X = \{0,1\}^{\binom{[3]}{2}}$ for short. By the lemma, we have

$$|G \mathbin{\backslash\backslash} X| = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

Observe that $|G| = 3! = 6$. Furthermore, consideration of the number of cycles gives

$$\begin{aligned} |X_{(1)(2)(3)}| &= 2^3 = 8, & |X_{(1\,2)(3)}| &= 2^2 = 4, \\ |X_{(1\,3)(2)}| &= 2^2 = 4, & |X_{(1)(2\,3)}| &= 2^2 = 4, \\ |X_{(1\,2\,3)}| &= 2^1 = 2, & |X_{(1\,3\,2)}| &= 2^1 = 2 \end{aligned}$$

and therefore

$$|G \mathbin{\backslash\backslash} X| = \frac{1}{|G|} \sum_{g \in G} |X_g| = \tfrac{1}{6}(8 + 4 + 4 + 4 + 2 + 2) = 24/6 = 4.$$

Exercise 2 Apply the Cauchy-Frobenius Lemma in order to determine

$$|S_n \mathbin{\backslash\backslash} \{0,1\}^{\binom{[n]}{2}}|$$

for $n = 4$.

Write $G = S_4$ and $X = \{0,1\}^{\binom{[4]}{2}}$ for short. Consider S_4 as a subgroup of S_X. Using the abreviation

a	b	c	d	e	f
$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$

we find the following induced permutation representation.

$\pi \in S_4$	$\bar\pi \in S_X$	$\mathrm{cn}(\bar\pi)$
id	$\mathrm{id} = (a)(b)(c)(d)(e)(f)$	6
$(1\,2\,3\,4)$	$\begin{pmatrix} a & b & c & d & e & f \\ d & e & a & f & b & c \end{pmatrix} = (a\,d\,f\,c)(b\,e)$	2
$(1\,2\,4\,3)$	$\begin{pmatrix} a & b & c & d & e & f \\ e & a & d & c & f & b \end{pmatrix} = (a\,e\,f\,b)(c\,d)$	2
$\vdots$		
$(1)(2\,3\,4)$	$\begin{pmatrix} a & b & c & d & e & f \\ b & c & a & f & d & e \end{pmatrix} = (a\,b\,c)(d\,e\,f)$	2
$\vdots$		
$(1\,2)(3)(4)$	$\begin{pmatrix} a & b & c & d & e & f \\ a & d & e & b & c & f \end{pmatrix} = (a)(b\,d)(c\,e)(f)$	4
$\vdots$		
$(1\,2)(3\,4)$	$\begin{pmatrix} a & b & c & d & e & f \\ a & e & d & c & b & f \end{pmatrix} = (a)(b\,e)(c\,d)(f)$	4

There are 6 permutations in S_4 of the shape $(\alpha\,\beta\,\gamma\,\delta)$, 8 permutations of the shape $(\alpha)(\beta\,\gamma\,\delta)$, 6 permutations of the shape $(\alpha\,\beta)(\gamma)(\delta)$, and 3 permutations of the shape $(\alpha\,\beta)(\gamma\,\delta)$. There is also the identity.

From the cycle numbers of the corresponding induced permutation representation according to the table above, we obtain

$$|G \mathbin{\backslash\backslash} X| = \tfrac{1}{24}(1 \cdot 2^6 + 3 \cdot 2^4 + 6 \cdot 2^4 + 6 \cdot 2^2 + 8 \cdot 2^2) = 11.$$

Exercise 3 How many necklaces with 5 beads in at most 3 colors are there?

Let C_5 act on $X := \{R,G,B\}^{[5]}$ by $\sigma f := f \circ \sigma^{-1}$. We have $C_5 = \langle \pi \rangle$ with $\pi = (1\,2\,3\,4\,5)$ and

$$\begin{aligned}
\pi^0 &= (1)(2)(3)(4)(5),\\
\pi^1 &= (1\,2\,3\,4\,5),\\
\pi^2 &= (1\,2\,3\,4\,5) \circ (1\,2\,3\,4\,5) = (1\,3\,5\,2\,4),\\
\pi^3 &= (1\,3\,5\,2\,4) \circ (1\,2\,3\,4\,5) = (1\,4\,2\,5\,3),\\
\pi^4 &= (1\,4\,2\,5\,3) \circ (1\,2\,3\,4\,5) = (1\,5\,4\,3\,2),\\
\pi^5 &= (1\,5\,4\,3\,2) \circ (1\,2\,3\,4\,5) = (1)(2)(3)(4)(5) = \pi^0.
\end{aligned}$$

It follows that

$$|C_5 \mathbin{\backslash\backslash} \{R,G,B\}^{[5]}| = \frac{1}{|C_5|} \sum_{\sigma \in C_5} |X_\sigma| = \tfrac{1}{5}(3^5 + 3 + 3 + 3 + 3) = 255/5 = 51.$$

Exercise 4 How many bracelets with 5 beads in at most 3 colors are there?

Let D_5 act on $X := \{R,G,B\}^{[5]}$ by $\sigma f := f \circ \sigma^{-1}$. We have $D_5 := \langle \pi, \tau \rangle$ with

$$\pi = (1\,2\,3\,4\,5) \qquad \text{and} \qquad \tau = (2\,5)(3\,4).$$

The elements of D_5 are therefore (see also Exercise 3):

$$\begin{aligned}
\pi^0 &= (1)(2)(3)(4)(5), & \tau\circ\pi^0 &= (2\,5)(3\,4)(1),\\
\pi^1 &= (1\,2\,3\,4\,5), & \tau\circ\pi^1 &= (2\,5)(3\,4)(1)\circ(1\,2\,3\,4\,5) = (1\,5)(2\,4)(3),\\
\pi^2 &= (1\,3\,5\,2\,4), & \tau\circ\pi^2 &= (2\,5)(3\,4)(1)\circ(1\,3\,5\,2\,4) = (1\,4)(2\,3)(5),\\
\pi^3 &= (1\,4\,2\,5\,3), & \tau\circ\pi^3 &= (2\,5)(3\,4)(1)\circ(1\,4\,2\,5\,3) = (1\,3)(2)(4\,5),\\
\pi^4 &= (1\,5\,4\,3\,2), & \tau\circ\pi^4 &= (2\,5)(3\,4)(1)\circ(1\,5\,4\,3\,2) = (1\,2)(3\,5)(4).
\end{aligned}$$

It follows that

$$|D_5 \setminus\!\!\setminus \{R,G,B\}^{[5]}| = \frac{1}{|D_5|}\sum_{\sigma\in D_5}|X_\sigma| = \tfrac{1}{10}(3^5 + 5\cdot 3^3 + 4\cdot 3) = 39.$$

Exercise 5 How many bracelets with 6 beads in at most 3 colors are there?

Let D_6 act on $X := \{R,G,B\}^{[6]}$ by $\sigma f := f\circ\sigma^{-1}$. We have $D_6 = \langle\pi,\tau\rangle$ with

$$\pi = (1\,2\,3\,4\,5\,6) \qquad \text{and} \qquad \tau = (1\,6)(2\,5)(3\,4).$$

The elements of D_6 are therefore

$$\begin{aligned}
\pi^0 &= (1)(2)(3)(4)(5)(6), & \tau\circ\pi^0 &= (1\,6)(2\,5)(3\,4),\\
\pi^1 &= (1\,2\,3\,4\,5\,6), & \tau\circ\pi^1 &= (1\,5)(2\,4)(3)(6),\\
\pi^2 &= (1\,3\,5)(2\,4\,6), & \tau\circ\pi^2 &= (1\,4)(2\,3)(5\,6),\\
\pi^3 &= (1\,4)(2\,5)(3\,6), & \tau\circ\pi^3 &= (1\,3)(2)(4\,6)(5),\\
\pi^4 &= (1\,5\,3)(2\,6\,4), & \tau\circ\pi^4 &= (1\,2)(3\,6)(4\,5),\\
\pi^5 &= (1\,6\,5\,4\,3\,2), & \tau\circ\pi^5 &= (1)(2\,6)(3\,5)(4).
\end{aligned}$$

It follows that

$$|D_6 \setminus\!\!\setminus X| = \frac{1}{|D_6|}\sum_{\sigma\in D_6}|X_\sigma| = \tfrac{1}{12}(1\cdot 3^6 + 3\cdot 3^4 + 4\cdot 3^3 + 2\cdot 3^2 + 2\cdot 3^1) = 92.$$