Manuel Kauers
RISC-Linz

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Algorithmic Combinatorics

Homework 6

Exercise 1 Show: Any equivalence relation $\sim$ on a set M can be induced by a suitably defined group action $G \times M \to M$.

We assume that M is countable. Let $\sim$ be an equivalence relation on M,

$$M/\!\sim \;= \{[m]_\sim : m \in M\}.$$

Let $[m_1]_\sim, [m_2]_\sim, \ldots$ be a (possibly finite) enumeration of $M/\!\sim$, and let M_i denote the cardinality of $[m_i]_\sim$ ($i \in \mathbb{N}$).
Consider the additive group $G = \mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2} \times \ldots$ (with $\mathbb{Z}_{M_i} := \mathbb{Z}$ if $M_i = \infty$). By writing

$$[m_i]_\sim := \begin{cases} \{m_{i,0}, m_{i,1}, \ldots, m_{i,M_i-1}\} & \text{if } M_i < \infty \\ \{\ldots, m_{i,-1}, m_{i,0}, m_{i,1}, \ldots\} & \text{if } M_i = \infty \end{cases},$$

and defining

$$(\pi_1, \pi_2, \ldots) m_{i,j} := \begin{cases} m_{i,j+\pi_i \bmod M_i} & \text{if } M_i < \infty \\ m_{i,j+\pi_i} & \text{if } M_i = \infty \end{cases}$$

for $(\pi_1, \pi_2, \ldots) \in G$ we obviously obtain a group action on M with $G \setminus\!\setminus M = M/\!\sim$.
That's it.

Exercise 2

a) How many lecklaces of six beads are there where exactly two beads are red, two beads are green, and two beads are blue?

b) Try to come up with a corresponding group action description of the problem.

c) How many necklaces of six beads in at most three colors are there?

Ad a) I found the following 16 ones:

1. RRGGBB	5. RRBGBG	9. RGRGBB	13. RBBRGG
2. RRGBGB	6. RRBGGB	10. RBRGGB	14. RBGRBG
3. RRGBBG	7. RGRBBG	11. RBRGBG	15. RBGRGB
4. RRBBGG	8. RGRBGB	12. RBRBGG	16. RGBRGB

Ad b) In a first step, we model the situation with distinct beads. Consider

$$\mathbb{Z}_6 \times S_6 \to S_6, \quad (\pi f)(n) \mapsto f(n + \pi \bmod 6).$$

Then $\mathbb{Z}_6 \setminus\setminus S_6$ is a group action description for the necklaces (strings up to cyclic rotation) with six different beads.

In the next step, we will identify pairs of beads in order to model the color. Say, the beads 1 and 2 are red, 3 and 4 are green, and 5 and 6 are blue. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ act on $\mathbb{Z}_6 \setminus\setminus S_6$ via

$$\begin{aligned} &G \times (\mathbb{Z}_6 \setminus\setminus S_6) \to \mathbb{Z}_6 \setminus\setminus S_6 \\ &((\pi_1, \pi_2, \pi_3)f)(n) := \begin{cases} f(\pi_1(n)) & n \in \{1,2\} \\ f(2 + \pi_2(n-2)) & n \in \{3,4\} \\ f(4 + \pi_3(n-4)) & n \in \{5,6\} \end{cases} \end{aligned}$$

where f is a representative of an orbit $[f]_\sim \in \mathbb{Z}_6 \setminus\setminus S_6$ and the elements of $\mathbb{Z}_2$ are considered as permutations of pairs in the obvious way. Observe that the definition is also independent of the choice of the representative, because the rotation of $\mathbb{Z}_6$ and swaping of G commute.

Also observe that we have defined a group action. For,

$$\begin{aligned} (1_G f)(n) &= (((0,0),(0,0),(0,0))f)(n) \\ (\sigma(\pi f))(n) &= \begin{cases} (\pi f)(\sigma_1(n)) & n \in \{1,2\} \\ 2 + (\pi f)(\sigma_2(n-2)) & n \in \{3,4\} \\ 4 + (\pi f)(\sigma_3(n-4)) & n \in \{5,6\} \end{cases} \\ &= \begin{cases} \pi_1(\sigma_1(n)) & n \in \{1,2\} \\ 2 + \pi_2(\sigma_2(n-2)) & n \in \{3,4\} \\ 4 + \pi_3(\sigma_3(n-4)) & n \in \{5,6\} \end{cases} \\ &= (\pi\sigma)f(n). \end{aligned}$$

We used that $\sigma_i(\{1,2\}) = \{1,2\}$ and that $\mathbb{Z}_2$ is commutative.

Ad c) I found the following 130 ones:

1. B B B B B B
2. B B B B B G
3. B B B B B R
4. B B B B G G
5. B B B B G R
6. B B B B R G
7. B B B B R R
8. B B B G B G
9. B B B G B R
10. B B B G G G
11. B B B G G R
12. B B B G R G
13. B B B G R R
14. B B B R B G
15. B B B R B R
16. B B B R G G
17. B B B R G R
18. B B B R R G
19. B B B R R R
20. B B G B B G
21. B B G B B R
22. B B G B G G
23. B B G B G R
24. B B G B R G
25. B B G B R R
26. B B G G B G
27. B B G G B R
28. B B G G G G
29. B B G G G R
30. B B G G R G
31. B B G G R R
32. B B G R B G
33. B B G R B R
34. B B G R G G
35. B B G R G R
36. B B G R R G
37. B B G R R R
38. B B R B B R
39. B B R B G G
40. B B R B G R
41. B B R B R G
42. B B R B R R
43. B B R G B G
44. B B R G B R
45. B B R G G G
46. B B R G G R
47. B B R G R G
48. B B R G R R
49. B B R R B G
50. B B R R B R
51. B B R R G G
52. B B R R G R
53. B B R R R G
54. B B R R R R
55. B G B G B G
56. B G B G B R
57. B G B G G G
58. B G B G G R
59. B G B G R G
60. B G B G R R
61. B G B R B R
62. B G B R G G
63. B G B R G R
64. B G B R R G
65. B G B R R R
66. B G G B G G
67. B G G B G R
68. B G G B R G
69. B G G B R R
70. B G G G B R
71. B G G G G G
72. B G G G G R
73. B G G G R G
74. B G G G R R
75. B G G R B R
76. B G G R G G
77. B G G R G R
78. B G G R R G
79. B G G R R R
80. B G R B G R
81. B G R B R G
82. B G R B R R
83. B G R G B R
84. B G R G G G
85. B G R G G R
86. B G R G R G
87. B G R G R R
88. B G R R B R
89. B G R R G G
90. B G R R G R
91. B G R R R G
92. B G R R R R

93. B R B R B R	103. B R G G R R	113. B R R R G G	123. G G R G R R
94. B R B R G G	104. B R G R G G	114. B R R R G R	124. G G R R G R
95. B R B R G R	105. B R G R G R	115. B R R R R G	125. G G R R R R
96. B R B R R G	106. B R G R R G	116. B R R R R R	126. G R G R G R
97. B R B R R R	107. B R G R R R	117. G G G G G G	127. G R G R R R
98. B R G B R G	108. B R R B R R	118. G G G G G R	128. G R R G R R
99. B R G B R R	109. B R R G G G	119. G G G G R R	129. G R R R R R
100. B R G G G G	110. B R R G G R	120. G G G R G R	130. R R R R R R
101. B R G G G R	111. B R R G R G	121. G G G R R R	
102. B R G G R G	112. B R R G R R	122. G G R G G R	

Exercise 3 Let Y be a G-set and X be a set. Define

$$G \times Y^X \to Y^X, \quad (\pi, f) \mapsto \pi f$$

by

$$(\pi f)(x) = \pi(f(x)) \qquad (x \in X).$$

Show that this is a group action.

Let $\pi_1, \pi_2 \in G, x \in X, f \in Y^X$. Then

$$(1_G f)(x) = 1_G(f(x)) = f(x)$$

and

$$(\pi_1(\pi_2(f)))(x) = \pi_1((\pi_2 f)(x)) = \pi_1(\pi_2(f(x))) = (\pi_1 \circ \pi_2)(f(x)).$$

Note that both times the last step makes use of the fact that X is a G-set.