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Algorithmic Combinatorics

Homework 5

Exercise 1 Find an elimination rule for $\Omega_{\geq} \dfrac{\lambda^{-s}}{(1-A\lambda)(1-B/\lambda)}$.

$$\begin{aligned}
\Omega_{\geq} \frac{\lambda^{-s}}{(1-\lambda A)(1-B/\lambda)} &= \Omega_{\geq} \lambda^{-s} \sum_{n=0}^{\infty} \lambda^n A^n \sum_{m=0}^{\infty} \lambda^{-m} B^m = \Omega_{\geq} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A^n B^m \lambda^{n-m-s} \\
&= \Omega_{\geq} \sum_{m=0}^{\infty} \sum_{n=m+s}^{\infty} A^n B^m \lambda^{n-m-s} = \Omega_{\geq} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A^{n+m+s} B^m \lambda^n \\
&= A^s \sum_{m=0}^{\infty} A^m B^m \sum_{n=0}^{\infty} A^n = \frac{A^s}{(1-AB)(1-A)}.
\end{aligned}$$

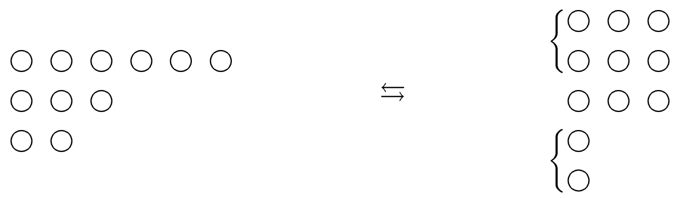
Exercise 2 Let $p_2(n,k)$ be the number of partitions of n into k parts with difference at least two. Prove: $\sum_{n=0}^{\infty} p_2(n,k) q^n = \frac{q^{k^2}}{(1-q)(1-q^2)\dots(1-q^k)}$.

$$\begin{aligned}
&\sum_{\substack{n_1,\dots,n_k \geq 1 \\ n_{i+1}-n_i \geq 2}} q^{n_1+\dots+n_k} = q^k \sum_{\substack{n_1,\dots,n_k \geq 0 \\ n_{i+1}-n_i \geq 2}} q^{n_1+\dots+n_k} \\
&= q^k \Omega_{\geq} \sum_{n_1,\dots,n_k \geq 0} \lambda_1^{n_2-n_1-2} \lambda_2^{n_3-n_2-2} \cdots \lambda_{k-1}^{n_k-n_{k-1}-2} q^{n_1+\dots+n_k} \\
&= q^k \Omega_{\geq} \frac{\lambda_1^{-2}\lambda_2^{-2}\cdots\lambda_{k-1}^{-2}}{(1-q\frac{1}{\lambda_1})(1-q\frac{\lambda_1}{\lambda_2})\cdots(1-q\frac{\lambda_{k-2}}{\lambda_{k-1}})(1-q\lambda_k)} \\
&\underset{B=q\lambda_{k-2}}{\overset{A=q}{=}} \quad q^k \frac{q^2}{(1-q)} \Omega_{\geq} \frac{\lambda_1^{-2}\cdots\lambda_{k-2}^{-2}}{(1-q\frac{1}{\lambda_1})(1-q\frac{\lambda_1}{\lambda_2})\cdots(1-q\frac{\lambda_{k-3}}{\lambda_{k-2}})(1-q^2\lambda_{k-2})} \\
&= \cdots = q^k \frac{q^2}{(1-q)} \frac{(q^2)^2}{(1-q^2)} \cdots \frac{(q^{k-2})^2}{(1-q^{k-2})} \Omega_{\geq} \frac{\lambda_1^{-2}}{(1-q\frac{1}{\lambda_1})(1-q^{k-1}\lambda_1)} \\
&= q^k \frac{q^2}{(1-q)} \frac{(q^2)^2}{(1-q^2)} \cdots \frac{(q^{k-2})^2}{(1-q^{k-2})} \frac{(q^{k-1})^2}{(1-q^{k-1})(1-q^k)} \\
&= \frac{q^k q^{2+4+6+\dots+2(k-1)}}{(1-q)\cdots(1-q^k)} = \frac{q^k q^{k(k-1)}}{(1-q)\cdots(1-q^k)} = \frac{q^{k^2}}{(1-q)\cdots(1-q^k)}
\end{aligned}$$

Exercise 3 For $n \in \mathbb{N}$, let $p_{\text{odd}}(n)$ be the number of partitions of n into odd parts and $p_{\text{dist}}(n)$ be the number of partitions of n into distinct parts. Give a combinatorial proof that $p_{\text{odd}}(n) = p_{\text{dist}}(n)$ for all $n \in \mathbb{N}$.

Given a partition $n = n_1 + n_2 + \cdots + n_k$ into odd parts, keep on replacing two equal parts $n_i = n_j$ by one part $n_i + n_j$. When this is no longer possible, the parts of the resulting partition are distinct.

Conversely, given a partition $n = n_1 + n_2 + \cdots + n_k$ into distinct parts, write $n_i = \mathrm{odd}_i \cdot 2^{m_i}$ and replace n_i by 2^{m_i} parts of size odd_i. The resulting partition consists of odd parts only.



Exercise 4 Find a recurrence for $p(n)$ and compute $p(721)$.

We are given

$$\sum_{n=1}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

By Euler's pentagonal number theorem, it follows that

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} p(n) q^n \cdot \sum_{n=-\infty}^{\infty} (-1)^n q^{r(3r-1)/2} \\ &= 1 + \sum_{n=1}^{\infty} (-1)^r q^{r(3r-1)/2} (1 + q^r). \end{aligned}$$

Defining $(a_n)_{n=0}^{\infty}$ by

$$\begin{aligned} \sum_{n=0}^{\infty} a_n q^n &= 1 + \sum_{n=1}^{\infty} (-1)^r q^{r(3r-1)/2} (1 + q^r) \\ &= 1 + \sum_{n=1}^{\infty} \Big((-1)^r q^{r(3r-1)/2} + (-1)^r q^{r(3r+1)/2} \Big) \end{aligned} \tag{1}$$

we get

$$1 = \sum_{n=0}^{\infty} p(n) q^n \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} p(k) a_{n-k}\, q^n.$$

Extracting the nth coefficient ($n > 0$) and observing $a_0 = 1$ gives the desired recurrence:

$$p(n) = \sum_{k=0}^{n-1} p(k) a_{n-k}.$$

As for the a_n, comparing coefficients in (1) gives the following "closed form":

$$a_n = \begin{cases} 1 & n = 0 \\ -1 & \text{if } (1 + \sqrt{1+24n}) \text{ or } (-1 + \sqrt{1+24n}) \text{ is an odd integer} \\ 1 & \text{if } (1 + \sqrt{1+24n}) \text{ or } (-1 + \sqrt{1+24n}) \text{ is an even integer} \\ 0 & \text{otherwise} \end{cases}$$

Observe that no natural number n can be written both as $n = r_1(3r_1 - 1)/2$ and $n = r_2(3r_2 + 1)/2$ for natural numbers r_1, r_2.

In addition, we need the starting value $p(1) = 1$.

Computing $p(721)$ with Mathematica takes 1.3s (on my machine) and gives the result

$$p(721) = 161\,061\,755\,750\,279\,477\,635\,534\,762.$$

Exercise 5 Find at least four group actions which have not been mentioned in the lecture.

a) Let G be the set of rotations around the origin in $\mathbb{R}^2$ and M be a circle around the origin. (obvious.)

b) Let $G = (\mathbb{R}, +)$, $M = \{-1, 0, 1\}$ and define $gm = \operatorname{sgn}(g \cdot m)$ for $g \in G$, $m \in M$. Then $1m = m$ $(m \in \mathbb{M})$ and

$$g_1(g_2m) = g_1 \operatorname{sgn}(g_2 \cdot m) = \operatorname{sgn}(g_1 \cdot \operatorname{sgn}(g_2 \cdot m)) = \operatorname{sgn}(g_1 g_2 \cdot m) = (g_1 g_2)m.$$

c) Let $G = (\mathbb{Z}, +)$ and $M = C^\infty([0,1])/\mathbb{R}[x]$. For $g \in G, m \in M$ define

$$gm = \begin{cases} \frac{\partial^g}{\partial x^g} m & \text{if } g < 0 \\ m & \text{if } g = 0 \\ \underbrace{\int \int \ldots \int}_{g \text{ times}} m & \text{if } g > 0 \end{cases}$$

d) Let M be any algebraic structure (semigroup, monoid, group, ring, ...) and G be the set of isomorphisms on M.

e) Let $G = (C([0,1]), +)$, $M = \mathbb{R}$ and define $gm = g(1) + m$. Then $1m = 0(1) + m = m$ and

$$g_1(g_2m) = g_1(g_2(1) + m) = g_1(1) + g_2(1) + m = (g_1 g_2)m$$