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Algorithmic Combinatorics
Homework 3

Exercise 1 Find nice recurrences for $s(n,k)$ and $c(n,k)$.

In order to enumerate all permutations π of $\{1,\dots,n\}$, we proceed recursively.

The first bunch of permutations leaves n fixed. Then (n) is one cycle of its own, and there are $k-1$ further cycles left for permuting the remaining $n-1$ elements. These gives $c(n-1,k-1)$ permutations.

Next we consider a permutation π of $\{1,\dots,n-1\}$ with k cycles. There are $c(n-1,k)$ permutations of this type. Let

$$\pi = (p_1\ p_2 \dots p_{c_1})(p_{c_1+1} \dots p_{c_2}) \dots\dots (p_{c_{k-1}+1} \dots p_{c_k})$$

be a representation of the permutation π. Note that $c_k = n-1$. Any extension of π to a permutation of $\{1,\dots,n\}$ also having exactly k cycles must have the form

$$\pi = (p_1 \dots) \dots (\dots p_i\ n\ p_{i+1} \dots) \dots (\dots p_{n-1})$$

for some $i = 1,\dots,n-1$. If p_i and p_{i+1} belong to different cycles, we may put n always to the cycle of p_i. This is without loss of generality because $(a_1\ a_2 \dots a_\ell) = (a_\ell\ a_1\ a_2 \dots a_{\ell-1})$. We have found that there are exactly $n-1$ ways to extend π to a permutation of $\{1,\dots,n\}$ leaving the number of cycles fixed.

Putting things together we obtain

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1) \qquad (n,k \geq 1)$$

as a recurrence for $c(n,k)$. Applying the definition $s(n,k) = (-1)^n c(n,k)$ directly gives the recurrence for $s(n,k)$:

$$s(n,k) = (1-n)s(n-1,k) + s(n-1,k-1) \qquad (n,k \geq 1).$$

Exercise 2 Prove: $\sum_{k=0}^{n} s(n,k)x^k = x^{\underline{n}}$.

Note that $s(n,k) = 0$ for $k > n$, so it suffices to show the claim for the generating function $G_n(x)$ of $s(n,k)$ in place of the finite sum. For $n = 0$ we have $G_0(x) = 1$. For $n > 0$ we have

$$G_n(x) = \sum_{k=0}^{\infty} s(n,k)x^k = (1-n)\sum_{k=0}^{\infty} s(n-1,k)x^k + \sum_{k=0}^{\infty} s(n-1,k-1)x^k$$
$$= (1-n)G_{n-1}(x) + \sum_{k=1}^{\infty} s(n-1,k-1)x^k = (1-n)G_{n-1}(x) + x\sum_{k=0}^{\infty} s(n-1,k)x^k$$
$$= (x-n+1)G_{n-1}(x).$$

This recurrence for $G_n(x)$ has the solution $G_n(x) = x^{\underline{n}}$.

Exercise 3 For $n, k \in \mathbb{N}$ define

$$J_n(k) := \{(a_1, \dots, a_n) \in \mathbb{N}^n : 0 \le a_i \le n - i \text{ and } \\ \text{exactly } k \text{ values of the } a_i \text{ are equal to } 0\}.$$

Prove $|J_n(k)| = c(n, k)$.

It suffices to show that $|J_n(k)|$ and $c(n, k)$ satisfy the same recurrence and the same initial conditions. As for the initial conditions, we find immediately

$$\begin{aligned} |J_0(0)| &= c(0,0) = 1, \\ |J_n(0)| &= c(n,0) = 0 \quad (n > 1), \\ |J_0(k)| &= c(0,k) = 0 \quad (k > 1). \end{aligned}$$

Now consider $J_n(k)$. Let $a = (a_1, \dots, a_n) \in J_n(k)$. If $a_1 = 0$ then there have to be $k-1$ further zeroes in the tail, and therefore $J_n(k)$ contains $|J_{n-1}(k-1)|$ tuples with $a_1 = 0$. Otherwise, if $a_1 \neq 0$, then there are $n-1$ possible values for a_1, and for each of these possibilities there are $|J_{n-1}(k)|$ tails. It follows that

$$|J_n(k)| = |J_{n-1}(k-1)| + (n-1)|J_{n-1}(k)| \qquad (n, k \ge 1),$$

which literally agrees with the recurrence for $c(n, k)$.

Exercise 4 Find a closed form expression for $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}$.

We have

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} \overset{(*)}{=} \prod_{k=1}^{n} \sum_{i=0}^{k-1} q^i = \prod_{k=1}^{n} \frac{q^k - 1}{q - 1} = \frac{1}{(q-1)^n} \prod_{k=1}^{n} (q^k - 1).$$

Defining the *q-analog* of $n \in \mathbb{N}$ as $[n]_q := \frac{q^n - 1}{q - 1}$, we can write the answer as $[n]_q!$ (*q-factorial*).

It remains to justify $(*)$. For $n = 1$ the assertion is evident $(1 = 1)$. Assume it holds for $n \in \mathbb{N}$. The elements of S_{n+1} can be created from the elements π of S_n by inserting $n+1$ at all possible positions of the word representation of π. For example, $(2\,1\,3)$ yields $(4\,2\,1\,3)$, $(2\,4\,1\,3)$, $(2\,1\,4\,3)$ and $(2\,1\,3\,4)$. If $\pi \in S_n$ has i inversions and $\pi' \in S_{n+1}$ is obtained from π by inserting $n+1$ at position $j \in [n+1]$, then π' has $i + (j-1)$ inversions.

It follows that

$$\begin{aligned} \sum_{\pi \in S_{n+1}} q^{\mathrm{inv}(\pi)} &= \Big(\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}\Big) + q\Big(\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}\Big) + \dots + q^n\Big(\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}\Big) \\ &= \Big(\sum_{k=0}^{n} q^k\Big) \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \Big(\sum_{k=0}^{n} q^k\Big) \prod_{k=1}^{n} \sum_{i=0}^{k-1} q^i = \prod_{k=1}^{n+1} \sum_{i=0}^{k-1} q^i. \end{aligned}$$

This completes the proof.

Exercise 5 For each number $k \in [0, n! - 1]$ $(n \in \mathbb{N})$ there exists a unique tuple $(\alpha_1, \dots, \alpha_{n-1}) \in [0,1] \times [0,2] \times \dots \times [0, n-1]$ such that

$$k = \sum_{i=1}^{n-1} \alpha_i i!.$$

The statement holds for $n = 1$. Suppose it holds for $n \in \mathbb{N}$. Observe the partition

$$[0, (n+1)! - 1] = [0, n! - 1] \,\dot{\cup}\, [n!, 2n! - 1] \,\dot{\cup} \ldots \dot{\cup}\, [nn!, (n+1)n! - 1].$$

For every $k \in [0, (n+1)! - 1]$ there is exactly one $i \in [1, n+1]$ such that $k \in [(i-1)n!, in! - 1]$. By induction hypothesis, the claim holds for $k - (i-1)n! \in [0, n! - 1]$, giving the unique tuple $(\pi_1, \ldots, \pi_{n-1})$. This extends to the unique tuple $(\pi_1, \ldots, \pi_{n-1}, i)$ for the original k.