Exercise 1 *Find a proof for*

$$\binom{n+x-1}{n} = \sum_{a \vdash n} \prod_{i=1}^{n} \frac{x^{a_i}}{i^{a_i} a_i!}$$

which is different from the one given in the lecture.

Solution

First observe that

$$\frac{n!}{\prod_{i=1}^{n} a_i! \prod_{i=1}^{n} i^{a_i}}$$

is the number of $\pi \in S_n$ having cycle type $a = (a_1, \ldots, a_n)$. This can be seen by the following combinatorial argument. Construct the cycle representation of any π with cycle type a starting with a permutation of n numbers,

$$\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-1}, \pi_n.$$

Next, cut out cycles according to the cycle type, giving

$$(\pi_1, \ldots, \pi_i)(\pi_{i+1}, \ldots, \pi_j) \ldots (\pi_k, \ldots, \pi_n).$$

This gives all permutations of cycle type a, but some of them have been counted too often. First, permuting cycles of the same length does not change the permutation. Hence the factor $1/\prod_{i=1}^{n} a_i!$. Secondly, every cycle of length i can be shifted to i different representations. This explains the factor $1/\prod_{i=1}^{n} i^{a_i}$.

Using this counting result, we can compute

$$\begin{aligned}
\sum_{a \vdash n} \prod_{i=1}^{n} \frac{x^{a_i}}{i^{a_i} a_i!} &= \frac{1}{n!} \sum_{a \vdash n} \frac{n!}{\prod_{i=1}^{n} a_i! \prod_{i=1}^{n} i^{a_i}} x^{\sum_{i=1}^{n} a_i} = \frac{1}{n!} \sum_{\pi \in S_n} x^{\mathrm{cn}(\pi)} \\
&= \frac{1}{n!} \sum_{k=0}^{n} c(n,k) x^k = \frac{1}{n!} x^{\overline{n}} = \frac{(n+x-1)^{\underline{n}}}{n!} = \binom{n+x-1}{n}.
\end{aligned}$$

The identity $\sum_{k=0}^{n} c(n,k) x^k = x^{\overline{n}}$ used in this calculation can be easily verified.

Exercise 2 *Let $H \times G$ act on Y^X. Prove*

$$|(H \times G) \setminus\!\setminus Y^X| = \frac{1}{|H||G|} \sum_{(h,g) \in H \times G} \prod_i |Y_{h^i}|^{a_i(\bar{g})}.$$

Solution

By the Cauchy-Frobenius-Lemma, we only need to show that the product on the right is equal to $|Y^X_{(h,g)}|$.

Consider $f \in Y^X_{(h,g)}$. For every $x \in X$, all values of f in the cycle of x in $\bar{g}$ are determined by a single value $f(x)$, because $hf(g^{-1}x) = f(x)$ forces $f(g^{-1}x) = h^{-1}f(x)$, and iteration determines the values on the entire cycle. If the cycle has length i, we end up with $h^{-i}f(x) = f(g^{-i}x) = f(x)$, so that $f(x) \in Y_{h^i}$.

On the other hand, any value $f(x) \in Y_{h^i}$ can be assigned to a canonical representative x of an i-cycle of $\bar{g}$ in order to define uniquely a function $f \in Y^X_{(h,g)}$.

Exercise 3 *Prove:* $|S_{[6]} \setminus\setminus \{0,1\}^{\binom{[6]}{2}}| = 156$ *using*

$$|S_{[\nu]} \setminus\setminus X| = \sum_{a \vdash \nu} \frac{(k+1)^{c_a}}{\prod_i a_i! i^{a_i}}$$

where $$c_a = \tfrac{1}{2}\sum_i i a_i^2 - \tfrac{1}{2}\sum_{i \text{ odd}} a_i + \sum_{r<s} a_r a_s \gcd(r,s)$$

as shown in the lecture.

Solution

a_1	a_2	a_3	a_4	a_5	a_6	c_a	$\prod_i a_i! i^{a_i}$	3^{c_a}	$3^{c_a}/\prod_i a_i! i^{a_i}$
6	0	0	0	0	0	15	720	32768	2048/45
4	1	0	0	0	0	11	48	2048	128/3
2	2	0	0	0	0	9	16	512	32
0	3	0	0	0	0	9	48	512	32/3
3	0	1	0	0	0	7	18	128	64/9
1	1	1	0	0	0	5	6	32	16/3
0	0	2	0	0	0	5	18	32	16/9
2	0	0	1	0	0	5	8	32	4
0	1	0	1	0	0	5	8	32	4
1	0	0	0	1	0	3	5	8	8/5
0	0	0	0	0	1	3	6	8	4/3
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Exercise 4 *Let G act on Y^X and $g \in G$. Let R be a ring with characteristic zero and $W : Y \to R$. Prove:*

$$\sum_{f \in {Y^X}_g} \prod_{x \in X} W(f(x)) = \prod_{i=1}^{|X|} \Big(\sum_{y \in Y} W(y)^i \Big)^{a_i(\bar{g})}.$$

Solution

$f \in {Y^X}_g$ if and only if f is fixed on all cycles of $\bar{g}$. Therefore, for every $f \in {Y^X}_g$, the product in the left hand side has the form $W(y_1)^{l_1} \cdots W(y_m)^{l_m}$ where y_i runs over the values of the cycles of $\bar{g}$ and l_i is the length of cycle i.

Every such product corresponds to one "distributive path" in the expansion of the right product. (The sum expresses the freedom to choose an independent value for each cycle.)