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Algorithmic Combinatorics

Homework 10

Exercise 1 Prove the following variant of the Cauchy-Frobenius Lemma:
Let X be a finite G-set, R a commutative ring such that $\mathbb{Q} \subseteq R$, and $w\colon X \to R$ be a constant on the G-orbits of X, i.e., $w(gx) = w(g'x)$ for all $x \in X$, $g, g' \in G$. Then for any set T being a complete set of representatives of $G \setminus\setminus M$ we have

$$\sum_{t\in T} w(t) = \frac{1}{|G|} \sum_{g\in G} \sum_{x\in X_g} w(x).$$

By $[A(x)]$ denote the indicator function that evaluates to 1 if $A(x)$ is true and to 0 otherwise. The we can reason:

$$\begin{aligned}\sum_{g\in G}\sum_{x\in X_g} w(x) &= \sum_{g\in G}\sum_{x\in X}[x\in X_g]w(x) = \sum_{x\in X}\sum_{g\in G}[x\in X_g]w(x)\\ &= \sum_{x\in X} w(x)|G_x| = \sum_{x\in X} w(x)\frac{|G|}{|G(x)|} = |G|\sum_{t\in T}\sum_{x\in G(t)} w(x)\frac{1}{|G(x)|}\\ &= |G|\sum_{t\in T} w(t) \underbrace{\sum_{x\in G(t)}\frac{1}{|G(t)|}}_{=1}.\end{aligned}$$

Exercise 2 Let G be a group, $H \leq G$. Show that

$$H \times G \to G, \qquad (h, g) \mapsto hg := h \circ g \circ h^{-1}$$

defines a group action of H on G.

First, $1g = 1 \circ g \circ 1^{-1} = g$ for all $g \in G$. Secondly, for all $h_1, h_2 \in H$ and $g \in G$ we have

$$\begin{aligned}h_1h_2g &= h_1(h_2 \circ g \circ h_2^{-1}) = h_1 \circ h_2 \circ g \circ h_2^{-1} \circ h_1^{-1} = (h_1 \circ h_2) \circ g \circ (h_1 \circ h_2)^{-1}\\ &= (h_1 \circ h_2)g.\end{aligned}$$

Exercise 3 Prove the following variant of the Cauchy-Frobenius Lemma:
Let X be a G-set, $H \leq G$ and T_G be a complete set of of representatives of the conjugacy classes of G. Then

$$|H \setminus\setminus X| = \frac{1}{|H|} \sum_{g\in T_G} |C^G(g) \cap H| \cdot |X_g|.$$

By $|X_g| = |X_{h \circ g \circ h^{-1}}|$ we can collaps the sum to

$$\sum_{g \in G} |X_g| = \sum_{g \in T_G} |C^G(g)| \cdot |X_g|.$$

Considering only elements of H, this argument gives

$$\sum_{h \in H} |X_h| = \sum_{g \in T_G} |C^G(g) \cap H| \cdot |X_g|.$$

The rest follows by applying the standard Cauchy-Frobenius Lemma to H and X.