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Algorithmic Combinatorics

Homework 1

Exercise 1 Find a rank function for Lists(n,k), i.e., a bijection $r\colon \text{Lists}(n,k) \to [n^{\underline{k}}]$ such that r and r^{-1} are easy to compute.

For simplicity, our bijection will be between Lists(n,k) and $\{0,1,\dots,n^{\underline{k}}-1\}$. It can be brought to the desired form by a few additional lines of code.

We propose the following algorithm for converting lists to numbers and vice versa. In our notation, l_i refers to the ith element of the list l.

```
function list2number(n, k, l)
  if k = 1 then
    return l_1 − 1
  else
    l' := [l_2, l_3, ..., l_k]
    for i from 2 to k do
      if l'_i > l_1 then l'_i := l'_i − 1
    return (l_1 − 1) · # Lists(n − 1, k − 1) + list2number(n − 1, k − 1, l')
```

The idea is to use a digit system whose base varies with the position of the digit. The first element of the list gives the highest "digit," and the tail has to be brought back into a form which allows recursion. The inverse is easily obtained as follows.

```
function number2list(n, k, index)
  if k = 1 then
    return [index + 1]
  else
    subindex = index mod # Lists(n − 1, k − 1)
    l_1 := 1 + (index quo # Lists(n − 1, k − 1))
    l' := number2list(n − 1, k − 1, subindex)
    for i from 1 to k − 1 do
      if l'_i ≥ l_1 then l'_i := l'_i + 1
    return [l_1, l'_1, l'_2, ..., l'_{k−1}]
```

Remark: An implementation of these algorithms in Maple is available at
http://www.risc.uni-linz.ac.at/mkauers/exercises/ .

Exercise 2 Give a combinatorial proof for the identity

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $n \in \mathbb{N}$ and x is an indeterminate.

We have to argue that $[x^k](1+x)^n = \binom{n}{k}$. Indeed,

$$(1+x)^n = \underbrace{(1+x)\cdot(1+x)\cdot\dots\cdot(1+x)}_{n \text{ factors}},$$

and on multiplying this out, we obtain 2^n terms from the 2^n path through this product (each factor admits a choice between 1 and x). A walk through the product gives x^k for exactly those paths which contain k x's and $n-k$ 1's. There are $|\operatorname{Sets}(n,k)| = \binom{n}{k}$ such paths.

Exercise 3 Show that the Printer procedure from the lecture defines a bijection between LabTrees(n) and $[n]^{[n-2]}$.

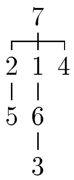
We have to show that for every $\pi \in [n]^{[n-2]}$ there exists exactly one tree T such that Printer(T) $= \pi$. We do so by giving an algorithm which computes this T from a given π. The following algorithm does that job.

function UnPrinter(n, π)
  // Assume $\pi = (\pi_1, \ldots, \pi_{n-2})$ with $\pi_i \in [n]$.
  $E := \{(n, \pi_{n-2})\}$ // first edge
  $\mathit{found} := \{\,\}$
  for i **from** 1 **to** $n-2$ **do**
    // create a new edge
    $e_1 := \pi_i$
    $e_2 := \min(\overline{\{\pi_i, \ldots, \pi_{n-2}\} \cup \mathit{found}})$ // complement wrt. $[n]$
    $\mathit{found} := \mathit{found} \cup \{e_2\}$
    $E := E \cup \{(e_1, e_2)\}$
  return $([n], E)$

Example: The tree from the lecture has given $\pi = (6\,7\,2\,7\,1)$. From this information the above algorithm reconstructs the tree defined by the edge $(7,1)$ and the edges $(6,a)$, $(7,b)$, $(2,c)$, $(7,d)$, $(1,e)$, where

$$\begin{aligned}
a &= \min\{\, k \in [7] : k \notin \{6,7,2,7,1\} \text{ and } k \notin \{\}\,\} &&= 3\\
b &= \min\{\, k \in [7] : k \notin \{7,2,7,1\} \text{ and } k \notin \{3\}\,\} &&= 4\\
c &= \min\{\, k \in [7] : k \notin \{2,7,1\} \text{ and } k \notin \{3,4\}\,\} &&= 5\\
d &= \min\{\, k \in [7] : k \notin \{7,1\} \text{ and } k \notin \{3,4,5\}\,\} &&= 2\\
e &= \min\{\, k \in [7] : k \notin \{1\} \text{ and } k \notin \{3,4,5,2\}\,\} &&= 6.
\end{aligned}$$

This gives the tree



known from the lecture.