What happened so far...

- **Recall: Discrete l'Hospital**
  Let (a_n), (b_n) be sequences, $b_n \to \infty$ as $n \to \infty$ where (b_n) is eventually monotonous. Then,
  $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\Delta a_n}{\Delta b_n}$$
  if the right hand side exists.
- **Application:** Limits of sequences involving sums
- **Examples:** (from the first talk, last semester)
  $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} k!}{n!} = 1, \qquad \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \binom{2k}{k}}{\binom{2n}{n}} = \tfrac{4}{3}, \qquad \lim_{n\to\infty} \frac{1}{\sqrt{n!}} \sum_{k=1}^{n} \sqrt{k!} = 1, \qquad \lim_{n\to\infty} n \cdot 2^{-n} \sum_{k=1}^{n} \frac{2^k}{k} = 2, \qquad \text{etc.}$$
- But isn't this too easy?

What about more interesting inputs?

- Do we find $\lim_{n\to\infty} \dfrac{n\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k}}{\mathrm{H}_n\, 2^n} = 2$?
- For humans, this is still very simple:

$$\lim_{n\to\infty} \frac{n\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k}}{\mathrm{H}_n\, 2^n} = \Big(\lim_{n\to\infty} \frac{\mathrm{H}_{2n}}{\mathrm{H}_n}\Big) \cdot \Big(\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{2^k}{k}}{2^n/n}\Big) \overset{\text{l'H}}{=} \Big(\lim_{n\to\infty} \frac{4n+3}{4n+2}\Big) \cdot \Big(\lim_{n\to\infty} \frac{2n}{n-1}\Big) = 2.$$

- Can a machine do that?
- (Suppose we can automatically detect monotonicity and going to infinity.)

What about more interesting inputs? II

- Let's see

$$\lim_{n\to\infty} \frac{n\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k}}{\mathrm{H}_n\, 2^n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{\left(\frac{3}{2} + 2n^2\,\mathrm{H}_{2n} + 3n\,\mathrm{H}_{2n} + \mathrm{H}_{2n} + 2n^2 + \frac{7}{2}n\right) \sum_{k=1}^{n} \frac{2^k}{k} + (4n^2+6n+2)2^n\,\mathrm{H}_{2n} + (4n+3)2^n}{(2n^2+3n+1)\,\mathrm{H}_n\, 2^n + (4n+1)2^n}$$

$$= \lim_{n\to\infty} \frac{p_1(n)\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k} + p_2(n) \sum_{k=1}^{n} \frac{2^k}{k} + p_3(n)\,\mathrm{H}_{2n}\, 2^n + p_4(n)2^n}{p_5(n)\,\mathrm{H}_n\, 2^n + p_6(n)2^n} \qquad \text{for some } p_i \text{ with } \deg p_i \leq 2$$

$$\overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{p_1(n)\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k} + p_2(n) \sum_{k=1}^{n} \frac{2^k}{k} + p_3(n)\,\mathrm{H}_{2n}\, 2^n + p_4(n)2^n}{p_5(n)\,\mathrm{H}_n\, 2^n + p_6(n)2^n} \qquad \text{for some } p_i \text{ with } \deg p_i \leq 5$$

$$\overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{p_1(n)\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k} + p_2(n) \sum_{k=1}^{n} \frac{2^k}{k} + p_3(n)\,\mathrm{H}_{2n}\, 2^n + p_4(n)2^n}{p_5(n)\,\mathrm{H}_n\, 2^n + p_6(n)2^n} \qquad \text{for some } p_i \text{ with } \deg p_i \leq 8$$

$$\overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{p_1(n)\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k} + p_2(n) \sum_{k=1}^{n} \frac{2^k}{k} + p_3(n)\,\mathrm{H}_{2n}\, 2^n + p_4(n)2^n}{p_5(n)\,\mathrm{H}_n\, 2^n + p_6(n)2^n} \qquad \text{for some } p_i \text{ with } \deg p_i \leq 11$$

- Continuation seems hopeless :-(

Limits of Rational Functions

- How do we see that
$$\lim_{n\to\infty}\frac{n^3+2n^2-n+5}{2n^3-7n^2+n+1}=\tfrac{1}{2} \quad ?$$
- There are *dominant terms* – the others don't contribute asymptotically:
$$\lim_{n\to\infty}\frac{n^3+2n^2-n+5}{2n^3-7n^2+n+1}=\lim_{n\to\infty}\frac{n^3}{2n^3}=\lim_{n\to\infty}\frac{1}{2}=\tfrac{1}{2}.$$
- Idea: Extend this technique to the multivariate case
- Define *dominant terms* $\mathrm{domT}(\cdot)$ such that
$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\mathrm{domT}(a_n)}{\mathrm{domT}(b_n)}$$
and $\mathrm{domT}(X)$ is "simpler" than X.

Dominant Terms

- **Definition** Let (a_n), (b_n) be sequences, and define

$$(a_n) \prec (b_n) \qquad :\Longleftrightarrow \qquad \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Say: (b_n) *grows faster* than (a_n).

- **Definition**

$$\operatorname{domT}(a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(r)}) = \max_{i=1}^{r} a_n^{(i)}$$

where $\max$ is taken with respect to $\prec$.

- **Example** $\operatorname{domT}(n + \mathrm{H}_n + n^2 + 1) = n^2$ because

$$\begin{aligned} \lim_{n\to\infty} n/\,\mathrm{H}_n = \infty \quad &\Longrightarrow \quad n \succ \mathrm{H}_n \\ \lim_{n\to\infty} n/n^2 = 0 \quad &\Longrightarrow \quad n^2 \succ n \\ \lim_{n\to\infty} n^2/1 = \infty \quad &\Longrightarrow \quad n^2 \succ 1. \end{aligned}$$

Dominant Terms II

- $\prec$ is not a strict order.
- What if $\max_i a_n^{(i)}$ doesn't exist?
- A more general **Definition**

$$\operatorname{domT}(a_n + b_n) := \begin{cases} a_n & \text{if } \lim_{n\to\infty} b_n/a_n = 0 \\ b_n & \text{if } \lim_{n\to\infty} a_n/b_n = 0 \\ (1+\frac{1}{\alpha})a_n & \text{if } \lim_{n\to\infty} a_n/b_n = \alpha \notin \{0,-1,\pm\infty\} \\ \mathrm{FAIL} & \text{otherwise} \end{cases}$$

  Recursive for $a_n^{(1)} + a_n^{(2)} + \cdots + a_n^{(r)}$

- It's clear that this definition also serves our purpose.

Towards an Algorithm

- INPUT: a_n
- OUTPUT: $\lim_{n\to\infty} a_n$, or FAIL.
- check special cases

if $a_n = p_n/q_n$ then
    $p_n := \mathrm{domT}(p_n)$
    $p_n := \mathrm{domT}(q_n)$
    if $q_n \to \infty$ then
        $\alpha := \lim_{n\to\infty} \frac{\Delta p_n}{\Delta q_n}$
    else
        $\alpha := \lim_{n\to\infty} p_n \Big/ \lim_{n\to\infty} q_n$
    fi
    return α
fi
return FAIL

Towards an Algorithm – FAQ

- In l'Hospital: How to decide monotonicity?

  — By restriction of admissible functions

- How to decide $a_n \to \infty$?

  — By applying a bag of tricks

- If there are several ways to write $a_n = p_n/q_n$?

  — Try them all, until one works

- Will this terminate?

  — No, use breadth first search!

- This has exponential complexity!

  — So what!

- Do you have implemented all that stuff?

  — Yes, in MAPLE

What about more interesting inputs? III

- Problem $\lim_{n\to\infty} \dfrac{n\,\mathrm{H}_{2n} \sum_{k=1}^{n} \frac{2^k}{k}}{\mathrm{H}_n\, 2^n}$ revisited.
- Step 1: l'Hospital because $\mathrm{H}_n\, 2^n \to \infty$ $(n \to \infty)$.

$$\frac{\left(\frac{3}{2} + 2n^2\,\mathrm{H}_{2n} + 3n\,\mathrm{H}_{2n} + \mathrm{H}_{2n} + 2n^2 + \frac{7}{2}n\right) \sum_{k=1}^{n} \frac{2^k}{k} + (4n^2 + 6n + 2)2^n\,\mathrm{H}_{2n} + (4n+3)2^n}{(2n^2 + 3n + 1)\,\mathrm{H}_n\, 2^n + (4n+1)2^n}$$

- Step 2: Reduction to dominant terms; cancellation

$$\frac{4\,n^2\,2^n\;\mathrm{H}_{2n}}{2\,n^2\,2^n\;\mathrm{H}_n} = \frac{2\,\mathrm{H}_{2n}}{\mathrm{H}_n}$$

- Step 3: Recursion: l'Hospital because $\mathrm{H}_n \to \infty$ $(n \to \infty)$

$$\frac{4n+3}{2n+1}$$

- Step 4: Reduction to dominant terms; cancellation

$$\frac{4n}{2n} = 2 \qquad \blacksquare$$

Examples Gallery

- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\Bigl(3^k\sum_{i=1}^k\frac{\mathrm{H}_i}{i}\Bigr)}{3^n\,\mathrm{H}_n^2}=\frac{3}{4}$
- $\displaystyle\lim_{n\to\infty}\frac{\mathrm{H}_n\sum_{k=1}^n\frac{2^k}{k}}{n\sum_{k=1}^n\frac{2^k\,\mathrm{H}_k}{k^2}}=1$
- $\displaystyle\lim_{n\to\infty}\frac{n^3\Bigl(\sum_{k=1}^n\frac{2^k}{k^2}\Bigr)^2}{2^n\sum_{k=1}^n\frac{2^k}{k}}=2$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\binom{2k}{k}2^k}{\binom{2n}{n}2^n}=\frac{8}{7}$
- $\displaystyle\lim_{n\to\infty}\frac{\mathrm{H}_n^2\sum_{k=1}^n\frac{\mathrm{H}_k}{k}}{n}=0$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\frac{\mathrm{H}_k}{k}}{\mathrm{H}_n^2}=\frac{1}{2}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\frac{k\,\mathrm{H}_k}{k^2+1}}{n+\mathrm{H}_{2n}^2}=0$
- $\displaystyle\lim_{n\to\infty}\frac{n!\sum_{k=1}^n\frac{2^k}{k}}{2^n\sum_{k=1}^n\frac{k!^2}{2^k}}=0$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\binom{3k}{k}\binom{5k}{k}}{\binom{3n}{n}\binom{5n}{n}}=\frac{84375}{83351}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\frac{2^k}{k^2}\binom{2k}{k}}{\Bigl(\sum_{k=1}^n\frac{2^k}{k^2}\Bigr)\sum_{k=1}^n\binom{2k}{k}}=\frac{3}{7}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^{2n}\frac{2^k k!}{k^2}}{\Bigl(\sum_{k=1}^n\frac{4^k}{k^2}\Bigr)\sum_{k=1}^{2n}k!}=\frac{3}{16}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\binom{2k}{k}\binom{3k}{k}}{\Bigl(\sum_{k=1}^n\binom{2k}{k}\Bigr)\sum_{k=1}^n\binom{3k}{k}}=\frac{69}{104}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\binom{2k}{k}^2}{\binom{2n}{n}^2}=\frac{16}{15}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^{2n}\frac{\mathrm{H}_k^2}{k}}{\mathrm{H}_n^3}=\frac{1}{3}$
- $\displaystyle\lim_{n\to\infty}\frac{\sum_{k=1}^n\Bigl(k^2\sum_{i=1}^k\frac{2^i}{i}\Bigr)}{2^n n}=4$

“Counterexamples” Gallery

- $\sum_{k=1}^{n} \frac{1}{k^2}$ – not of the form p_n/q_n
- $5/\sum_{k=1}^{n} \frac{1}{k^2}$ – $q_n \not\to \infty$
- $\dfrac{n \sum_{k=1}^{n} \frac{\mathrm{H}_k}{k}}{\mathrm{H}_n}$ – does not terminate:

$$\lim_{n\to\infty} \frac{n \sum_{k=1}^{n} \frac{\mathrm{H}_k}{k}}{\mathrm{H}_n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \Big(n \sum_{k=1}^{n} \frac{\mathrm{H}_k}{k} + \sum_{k=1}^{n} \frac{\mathrm{H}_k}{k} + n\,\mathrm{H}_n + \mathrm{H}_n + 1 \Big)$$

What is the dominant term?

What now?

- Short term:
  - Some hacking to improve the implementation
  - Perhaps AI-techniques for searchtree organisation
  - Comprehensive procedure for $a_n \to \infty$
  - Extended input class

- Medium- and Longterm:
  - Find more systematic approaches...
  - Collect interesting examples