

Schützenberger's factorization on q -stuffle Hopf algebraC. Bui[‡], G. H. E. Duchamp[#], Hoang Ngoc Minh^{◇,‡}[‡]Huế University - College of sciences, 77 - Nguyen Hue street - Huế city, Việt Nam[#]Institut Galilée, LIPN - UMR 7030, CNRS - Université Paris 13, F-93430 Villetaneuse, France,[◇]Université Lille II, 1, Place Déliot, 59024 Lille, France

Schützenberger's monoidal factorization [9] has been introduced and plays a central role in the renormalization [7] of associators which are formal power series in non commutative variables¹. The coefficients of these power series are polynomial at positive integral multi-indices of Riemann's zêta function² [5, 10] and they satisfy quadratic relations [1] which can be explained through Lyndon words. These relations can be obtained by identification of the local coordinates on a bridge equation connecting the Cauchy and Hadamard algebras of polylogarithmic functions and use the factorizations of the non commutative generating series of polylogarithms [6] and of harmonic sums [7]. This equation is mainly a consequence of the double isomorphy between these structures to respectively the shuffle [6] and stuffle [3] algebras both admitting the Lyndon words as a transcendence basis.

Symbolic computation allows us to introduce a formal variable q in order to better understand the mechanisms of the shuffle and to obtain algorithms on stuffles. We will then examine the q -stuffle interpolating between the shuffle [9], stuffle [8] and minus-stuffle [3]. In particular, we will give an effective construction of pair of bases in duality. It uses essentially an adapted version of the Eulerian projector in order to obtain the primitive elements of the q -stuffle Hopf algebra and they are obtained thanks to the computation of the logarithm of the diagonal series. This study completes the treatment for the stuffle [7] and boils down to the shuffle [9].

More precisely, let $Y = \{y_s\}_{s \geq 1}$ be an alphabet with the total order $y_1 > y_2 > \dots$. Let also \mathbf{k} be a unitary \mathbb{Q} -algebra containing q . One defines the q -stuffle, or its dual co-product, as follows, for any $y_s, y_t \in Y$ and $u, v \in Y^*$,

$$u \sqcup_q 1_{Y^*} = 1_{Y^*} \sqcup_q u = u \quad \text{and} \quad y_s u \sqcup_q y_t v = y_s(u \sqcup_q y_t v) + y_t(y_s u \sqcup_q v) + q y_{s+t}(u \sqcup_q v), \quad (1)$$

$$\Delta_{\sqcup_q}(1_{Y^*}) = 1_{Y^*} \otimes 1_{Y^*} \quad \text{and} \quad \Delta_{\sqcup_q}(y_s) = y_s \otimes 1_{Y^*} + 1_{Y^*} \otimes y_s + q \sum_{s_1+s_2=s} y_{s_1} \otimes y_{s_2}. \quad (2)$$

This product is commutative, associative and unital. With the co-unit defined by, $\epsilon(P) = \langle P \mid 1_{Y^*} \rangle$, for $P \in \mathbf{k}\langle Y \rangle$, one gets $\mathcal{H}_{\sqcup_q} = (\mathbf{k}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup_q}, \epsilon)$ and $\mathcal{H}_{\sqcup_q}^\vee = (\mathbf{k}\langle Y \rangle, \sqcup_q, 1_{Y^*}, \Delta_{\text{conc}}, \epsilon)$ which are mutually dual bialgebras and, in fact, Hopf algebras because they are \mathbb{N} -graded by the weight.

Group-like elements, redefined below, form a group for which the log-exp correspondence is explained by as follows

Lemma 1 (q -extended Friedrichs criterium) *Let $S \in \mathbf{k}\langle Y \rangle$ (for 2., we suppose in addition that $\langle S \mid 1_{Y^*} \rangle = 1$).*

1. S is primitive, i.e. $\Delta_{\sqcup_q} S = S \otimes 1_{Y^*} + 1_{Y^*} \otimes S$, if and only if, for any $u, v \in Y^+$, $\langle S \mid u \sqcup_q v \rangle = 0$.
2. S is group-like, i.e. $\Delta_{\sqcup_q} S = S \otimes S$, if and only if, for any $u, v \in Y^+$, $\langle S \mid u \sqcup_q v \rangle = \langle S \mid u \rangle \langle S \mid v \rangle$.
3. S is group-like if and only if $\log S$ is primitive.

Proposition 1 *Let $\mathcal{D}_Y = \sum_{w \in Y^*} w \otimes w$ be the diagonal series over Y . Then*

1. $\log \mathcal{D}_Y = \sum_{w \in Y^+} w \otimes \pi_1(w)$, where $\pi_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w \mid u_1 \sqcup_q \dots \sqcup_q u_k \rangle u_1 \dots u_k$.
2. For any $w \in Y^*$, we have $w = \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^+} \langle w \mid u_1 \sqcup_q \dots \sqcup_q u_k \rangle \pi_1(u_1) \dots \pi_1(u_k)$.

¹These associators were introduced in quantum field theory by Drinfel'd and the universal associator, i.e. Φ_{KZ} , was obtained with explicit coefficients which are polyzêtas and regularized polyzêtas [5].

²These values are usually abbreviated MZV's by Zagier [10] and are also called polyzêtas by Cartier [1].

Let $\mathcal{P} = \{P \in \mathbb{Q}\langle Y \rangle \mid \Delta_{\sqcup_q} P = P \otimes 1 + 1 \otimes P\}$ be the set of primitive polynomials. Since, in virtue of $\Delta_{\sqcup_q} \pi_1(w) = \pi_1(w) \otimes 1 + 1 \otimes \pi_1(w)$, $\text{Im}(\pi_1) \subseteq \mathcal{P}$, we can state the following

Theorem 1 ([2]) 1. Let $\{\Pi_l\}_{l \in \mathcal{L}_{yn}Y}$ be defined by, for any $y_k \in Y$, $\Pi_{y_k} = \pi_1(y_k)$ and for any $l \in \mathcal{L}_{yn}X$ of standard factorization $l = (s, r)$, $\Pi_l = [\Pi_s, \Pi_r]$. Then $\{\Pi_l\}_{l \in \mathcal{L}_{yn}Y}$ forms a basis of \mathcal{P} .

2. Let $\{\Pi_w\}_{w \in Y^*}$ be defined by, for any $w \in Y^*$ such that $w = l_1^{i_1} \dots l_k^{i_k}$, $l_1 > \dots > l_k$, $l_1, \dots, l_k \in \mathcal{L}_{yn}Y$, $\Pi_w = \Pi_{l_1}^{i_1} \dots \Pi_{l_k}^{i_k}$. Then $\{\Pi_w\}_{w \in Y^*}$ forms a basis of $\mathbf{k}\langle Y \rangle$.

3. Let $\{\Sigma_w\}_{w \in Y^*}$ be the family of the quasi-shuffle algebra obtained by duality with $\{\Pi_w\}_{w \in Y^*}$. Then $\{\Sigma_w\}_{w \in Y^*}$ generates freely the quasi-shuffle algebra.

4. The family $\{\Sigma_l\}_{l \in \mathcal{L}_{yn}Y}$ forms a transcendence basis of $(\mathbf{k}\langle Y \rangle, \sqcup_q)$.

We now give formulas which permit to compute the basis $\{\Sigma_w\}_{w \in Y^*}$ without inverting a huge Gram matrix.

Theorem 2 (q -extended Schützenberger's factorization, [2]) 1. For any $y \in Y$, $\Sigma_y = y$.

2. For any $y_{s_1} \dots y_{s_k} \in \mathcal{L}_{yn}X$, $\Sigma_{y_{s_1} \dots y_{s_k}} = \sum_{\substack{\{s'_1, \dots, s'_i\} \subset \{s_1, \dots, s_k\}, l_1 \geq \dots \geq l_n \in \mathcal{L}_{yn}Y \\ (y_{s_1} \dots y_{s_k}) \stackrel{*}{\leftarrow} (y_{s'_1}, \dots, y_{s'_n}, l_1, \dots, l_n)}} \frac{q^{i-1}}{i!} y_{s'_1 + \dots + s'_i} \Sigma_{l_1 \dots l_n}.$

3. For any $w = l_1^{i_1} \dots l_k^{i_k}$, with $l_1, \dots, l_k \in \mathcal{L}_{yn}Y$ and $l_1 > \dots > l_k$, $\Sigma_w = \frac{\Sigma_{l_1}^{\sqcup_q i_1} \sqcup_q \dots \sqcup_q \Sigma_{l_k}^{\sqcup_q i_k}}{i_1! \dots i_k!}.$

4. $\mathcal{D}_Y = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}_{yn}Y} \exp(\Sigma_l \otimes \Pi_l).$

Theorems 1.1 and 2.2 are based mainly on respectively the logarithm of the diagonal series \mathcal{D}_Y and the standard sequences [9, 2] and lead to simplified algorithms getting bases in duality as shown in the following

Example 1

$$\begin{aligned} \Pi_{y_2} &= y_2 - \frac{q}{2} y_1^2, \\ \Pi_{y_2 y_1} &= y_2 y_1 - y_1 y_2, \\ \Pi_{y_3 y_1 y_2} &= y_3 y_1 y_2 - \frac{q}{2} y_3 y_1^2 - q y_2 y_1^2 y_2 + \frac{q^2}{4} y_2 y_1^4 - y_1 y_3 y_2 + \frac{q}{2} y_1 y_3 y_1^2 + \frac{q}{2} y_1^2 y_2^2 - \frac{q^2}{2} y_1^2 y_2 y_1^2 - y_2 y_3 y_1 \\ &\quad + \frac{q}{2} y_2^2 y_1^2 + y_2 y_1 y_3 + \frac{q}{2} y_1^2 y_3 y_1 - \frac{q}{2} y_1^3 y_3 + \frac{q^2}{4} y_1^4 y_2, \\ \Pi_{y_3 y_1 y_2 y_1} &= y_3 y_1 y_2 y_1 - y_3 y_1^2 y_2 - \frac{q}{2} y_2 y_1^2 y_2 y_1 - y_1 y_3 y_2 y_1 + y_1 y_3 y_1 y_2 + \frac{q}{2} y_1^2 y_2^2 y_1 - \frac{q}{2} y_1^2 y_2 y_1 y_2 - y_2 y_1 y_3 y_1 \\ &\quad + \frac{q}{2} y_2 y_1 y_2 y_1^2 + y_2 y_1^2 y_3 + y_1 y_2 y_3 y_1 - \frac{q}{2} y_1 y_2^2 y_1^2 - y_1 y_2 y_1 y_3 + \frac{q}{2} y_1 y_2 y_1^2 y_2, \\ \Sigma_{y_2} &= y_2, \\ \Sigma_{y_2 y_1} &= y_2 y_1 + \frac{q}{2} y_3, \\ \Sigma_{y_3 y_2 y_1} &= y_3 y_1 y_2 + y_3 y_2 y_1 + q y_3^2 + \frac{q}{2} y_4 y_2 + \frac{q^2}{3} y_6 + \frac{q}{2} y_5 y_1, \\ \Sigma_{y_3 y_1 y_2 y_1} &= 2 y_3 y_2 y_1^2 + q y_3 y_2^2 + y_3 y_1 y_2 y_1 + \frac{3q}{2} y_3^2 y_1 + \frac{q}{2} y_3 y_1 y_3 + \frac{q^2}{2} y_3 y_4 + \frac{q}{2} y_4 y_2 y_1 + \frac{q^2}{4} y_4 y_3 + q y_5 y_1^2 + \frac{q^2}{2} y_5 y_2 + \frac{q^2}{2} y_6 y_1 + \frac{q^3}{8} y_7. \end{aligned}$$

In conclusion, since the pioneering works of Schützenberger and Reutenauer [9], the question of computing bases in duality (maybe at the cost of a more involved procedure, but without inverting a Gram matrix) remained open in the case of cocommutative deformations of the shuffle product. We have given such a procedure allowing a great simplification for an interpolation between shuffle and stuffle. In the next framework, this product will be continuously deformed, in the most general way while remaining commutative [4].

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