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Problem 68-17

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It is to be noted that for the case  $e = 0, n = b + 2c + 3d$ , the end binomial coefficient reduces to  $\binom{-1}{-1}$  which is to be taken as 1.

*Problem 68-17, A Definite Integral*, by B. F. LOGAN, C. L. MALLOWS and L. A. SHEPP (Bell Telephone Laboratories).

Evaluate the integral

$$I = \int_0^\infty e^{-w/2} \sqrt{w} \, du,$$

where

$$w = \frac{u}{1 - e^{-u}}.$$

The integral arose in a probability problem.

Solution by WILLIAM B. JORDAN (G. E. Knolls Atomic Power Laboratory).

Letting  $w = u + v$ , it follows that

$$\begin{aligned} e^u &= w/v, & we^{-w} &= ve^{-v}, \\ du &= dw/w - dv/v, & u]_0^\infty &= w]_1^\infty = v]_1^0. \end{aligned}$$

Whence,

$$I = \int_{u=0}^\infty \{we^{-w}\}^{1/2} \left\{ \frac{dw}{w} - \frac{dv}{v} \right\}$$

or

$$\begin{aligned} I &= \int_1^\infty \{we^{-w}\}^{1/2} \frac{dw}{w} - \int_1^0 \{ve^{-v}\}^{1/2} \frac{dv}{v} \\ &= \int_0^\infty t^{-1/2} e^{-t/2} \, dt = \sqrt{2\pi}. \end{aligned}$$

M. L. GLASSER (Battelle Memorial Institute) obtains the following generalization by means of the generating function for the generalized Laguerre polynomials:

$$\int_0^\infty \{ye^{-y}\}^v \, dx = \int_0^\infty \{ye^{-y}\}^v \, dy, \quad v > 0,$$

where  $y = x/(e^x - 1)$  or  $x/(1 - e^{-x})$ .

In the solution by the proposers, it was noted that Logan and Shepp obtained the result  $(2\pi)^{-1/2} I = 1$  as the expression of the probabilistic fact that a standard Wiener process is almost certain to meet a boundary curve. Additionally, Mallows obtains the still further generalization:

Suppose (i)  $h(x)$  and its derivative  $h'(x)$  are positive and monotone for  $-\infty < x < \infty$ , with  $h(x) = h(-x) + x$  and  $h(-\infty) = 0$ . Also suppose (ii)  $g(w)$  (defined for  $0 \leq w < \infty$ ) satisfies  $g(h(x)) \equiv g(h(-x))$  for  $-\infty < x < \infty$ . Then  $\int_0^\infty g(h(x)) dx = \int_0^\infty g(w) dw$  (with slight additional generality, for any  $f$  we have  $\int_0^\infty f(g(h(x))) dx = \int_0^\infty f(g(w)) dw$ ). The proof is straightforward. In the present case we have  $g(w) = (we^{-w})^{1/2}$ ,  $h(x) = x/(1 - e^{-x})$ .

In general, if the function  $g(h(x))$  is given, it may not be easy to see how to choose  $h$  so that (i) and (ii) are satisfied. If  $h$  is given satisfying (i) (there are many such functions; it is only necessary that  $h'(x) - \frac{1}{2}$  is odd and monotone, with  $h(-\infty) = 0$ ), then it is easy to construct many functions  $g(w)$  satisfying (ii). Any function of  $|h^{-1}(w)|$  will do; alternatively we may write  $g(w) = G(w, w - h^{-1}(w))$  where  $G(w, w')$  is any symmetric function. (Then  $g(w) = G(h(x), h(-x))$ , where  $w = h(x)$ .) In the present case  $G(w, w') = (|w - w'| \min(w, w')/ww')^{1/2}$ .