RECURRENCES FOR PERMUTATIONS WITH LONG INCREASING SUBSEQUENCES

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ABSTRACT. We prove two simple bivariate recurrences for the number of permutations with a long increasing subsequence. The two recurrences imply D-finiteness of the sequence in a certain range. As a consequence, we also obtain a proof of a conjecture posed by Kauers and Koutschan in 2023.

1. INTRODUCTION

Let

$$a_{n,k} := \#\{\sigma \in S_n \mid \mathrm{LIS}(\sigma) = k\},$$

where $\mathrm{LIS}(\sigma)$ denotes the length of the longest increasing subsequence of σ. We use the convention that $a_{n,k} = 0$ when $k < 1$ or $k > n$. The resulting triangular array appears in the OEIS as A047874 and starts like

$$\begin{array}{ccccccc} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ 1 & 13 & 9 & 1 & & & \\ 1 & 41 & 61 & 16 & 1 & & \\ 1 & 131 & 381 & 181 & 25 & 1 & \\ 1 & 428 & 2332 & 1821 & 421 & 31 & 1 \end{array}.$$

Gessel [1] showed that every column of this array is D-finite, i.e., for every fixed k, the sequence $(a_{n,k})_{n=0}^{\infty}$ satisfies a linear recurrence with polynomial coefficients in n. For every fixed n, the sequence $(a_{n,k})_{k=0}^{\infty}$ is finite and therefore trivially D-finite. However, both results combined are not enough to conclude that $a_{n,k}$ is D-finite as a bivariate sequence. Recall [2] that this requires a recurrence of the form

$$p_0(n,k)a_{n,k} + p_1(n,k)a_{n+1,k} + \cdots + p_r(n,k)a_{n+r,k} = 0$$

with certain polynomials $p_0, \ldots, p_r$, as well as a recurrence of the form

$$q_0(n,k)a_{n,k} + q_1(n,k)a_{n,k+1} + \cdots + q_s(n,k)a_{n,k+s} = 0$$

with certain polynomials $q_0, \ldots, q_s$. Rather than allowing each column (or row) to have its own recurrence, there must be a uniform recurrence that works for all columns (or rows) simultaneously.

There is no reason to expect that such recurrences exist for the sequence $a_{n,k}$, and we believe that there are no such recurrences that hold for all n and k. The main result of this note is that the bivariate sequence does become D-finite if we restrict its domain to the sector $n/2 \le k \le n$. Here and throughout the paper, we use the term "D-finite inside a region R" in the sense of "There exists a D-finite array that coincides with the original array inside R".

Theorem 1. *The two recurrences*

$$a_{n+1,k} + n^2 a_{n-1,k} = a_{n,k+1} + (2n-1)a_{n,k} + a_{n,k-1} \tag{1}$$

and

$$(n+1)^2(a_{n,k} + a_{n,k+1}) = (n-k+1)a_{n+1,k} + (n+k+1)a_{n+1,k+1} \tag{2}$$

are valid in the region $n/2 \le k$.

The recurrences announced in Theorem 1 and proved below both involve shifts with respect to both n and k. Recurrences involving only shifts in one of the directions, as required by the definition of D-finiteness, can be easily obtained from them by computer algebra. For example, if `basis` denotes a list containing the two operators corresponding to the two recurrences stated in the theorem, then typing

```
FindRelation[OreGroebnerBasis[basis], Support -> Table[S[n]^i, {i, 0, 4}]]
```

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and

```
FindRelation[OreGroebnerBasis[basis], Support -> Table[S[k]^i, {i, 0, 4}]]
```

in Koutschan's Mathematica package HolonomicFunctions [5] produces the desired recurrences effortlessly. We do not reproduce them here.

Similarly, we obtain a proof of Conjecture 17 in [4] as a corollary of Theorem 1. Setting

$$b_{n,k} := \#\{\sigma \in S_n \mid \mathrm{LIS}(\sigma) \geq k\} = \sum_{i=k}^{\infty} a_{n,i},$$

Conjecture 17 concerns the sequence $b_{2n,n}$, whose nth term is the number of permutations of length $2n$ containing an increasing subsequence of length n. This sequence appears as A269021 in the OEIS, and based on the terms given there, Kauers and Koutschan conjectured that $b_{2n,n}$ is D-finite, satisfying a recurrence of order 4.

Using Koutschan's package, a recurrence for $b_{2n,n}$ can be derived from Theorem 1 by typing the following commands:

```
G := OreGroebnerBasis[basis];
Q := CreativeTelescoping[G, S[k]-1, {S[n]}][[2,1]];
B := DFiniteOreAction[G, Q];
FindRelation[B, Support -> Table[S[n]^(2i)S[k]^i, {i,0,4}]]
```

Up to normalization, this produces precisely the recurrence stated in Conjecture 17 of [4].

Our proof of Theorem 1 is based on the Robinson–Schensted correspondence. It expresses $a_{n,k}$ as a sum of squares of dimensions of irreducible representations indexed by partitions with first row k. In the range $k \geq n/2$, these partitions have a sufficiently long first row that the ordinary branching operators and the Murnaghan–Nakayama rim-hook operators take a particularly simple form. Combining these relations with character orthogonality will give (1) and (2).

2. Large scale calculation of $a_{n,k}$

The recurrences in (1) and (2) were found using guessing [3] from the first few sequence terms. In order to efficiently compute enough terms of the array $a_{n,k}$ to find these recurrences, we use Gessel's determinantal formula [1, Section 7] for the cumulative counts

$$u_k(n) := a_{n,1} + \cdots + a_{n,k}.$$

It states that (after suitable transformation of the original formula)

$$\sum_{n\geq 0} u_k(n) \frac{x^n}{(n!)^2} = \det\left(x^{(i-j)/2} I_{i-j}(2\sqrt{x})\right)_{i,j=1,\ldots,k}, \tag{3}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind, and the matrix entries have nice series expansions

$$x^{s/2} I_s(2\sqrt{x}) = \begin{cases} x^s \sum_{m\geq 0} \frac{x^m}{m!(m+s)!} & s \geq 0, \\ \sum_{m\geq 0} \frac{x^m}{m!(m-s)!} & s < 0. \end{cases}$$

To calculate $u_k(n)$ for n, k up to N and K utilizing (3) requires computing the leading principal minors of a $K \times K$ matrix whose entries are truncated power series in x of degree N. This is done by ordinary Gaussian elimination without pivoting. In order to prevent large integer arithmetic, we do the calculations modulo many primes (with each prime larger than $N + K$ and their product larger than $N!$) and then use the Chinese remainder theorem to recover $u_k(n)$. With this approach and naive truncated power series arithmetic, we can compute $u_k(n)$ for n, k up to N and K in time $O(N^2K^3)$ per prime (and therefore $O(N^3K^3 \log N)$ in total using fixed-word-size primes). Finally, we recover $a_{n,k}$ from $u_k(n)$ by taking differences.

The actual computation for $N = 300, K = 200$ took about 1 hour on an i5-12400 desktop computer, and produced enough data to guess the recurrences (1) and (2). For further investigation of the array $a_{n,k}$ (including guessing more complicated recurrences), we have also computed $u_k(n) \mod 2^{30} + 3$ for n, k up to $N = K = 1200$, which took about 1 day on the same computer.

It turns out that on the other side of the diagonal $k = n/2$, the array $a_{n,k}$ also satisfies bivariate recurrences, but they are more complicated. Experimentally we have found recurrences with higher order

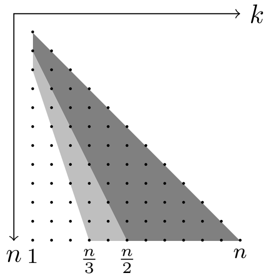

Figure 1. The triangular array $a_{n,k}$ is proven to be D-finite in the region marked in dark gray. In addition, we have a system of guessed recurrences which appear to hold for the sequence in the light gray region.

and degree for the region $n/3 < k < n/2$, the shortest among them is

$$
\begin{aligned}
&(n+1)^2(n+2)^2\left(\left(-2k^2+k+3\right)a_{n,k}-\left(2k^2+k-3\right)a_{n,k+1}\right.\\
&\qquad\left.+(2k+1)a_{n,k-1}+(1-2k)a_{n,k+2}\right)\\
&+(n+2)^2\left((-6+k+5k^2-2k^3-6n-kn+6k^2n)a_{n+1,k}\right.\\
&\qquad+(-6-k+5k^2+2k^3-6n+kn+6k^2n)a_{n+1,k+1}\\
&\qquad\left.-(1+2k)(4-3k+2n)a_{n+1,k-1}-(1-2k)(4+3k+2n)a_{n+1,k+2}\right)\\
&+\left((4n-2)k^3-(n^2+8n+7)k\right)\left(a_{n+2,k}-a_{n+2,k+1}\right)\\
&+\left(-(6n^2+22n+19)k^2+(3n^2+9n+6)\right)\left(a_{n+2,k}+a_{n+2,k+1}\right)\\
&+(2k+1)(k-n-3)(2k-n-2)a_{n+2,k-1}\\
&-(2k-1)(k+n+3)(2k+n+2)a_{n+2,k+2}\\
&+k(2k-1)(k+n+3)a_{n+3,k+1}-k(2k+1)(k-n-3)a_{n+3,k}=0,
\end{aligned}
$$

Interestingly enough, these complicated recurrences also appear to hold for the region $k \geq n/2$, and we have checked by direct reduction that they lie in the (left) ideal generated by (1) and (2). We suspect that for every $r = 2, 3, \ldots$, the sequence $a_{n,k}$ is D-finite in the range $k \geq n/r$. Note that even if this is true, it would not imply that $a_{n,k}$ is D-finite on the whole range $k \geq 0$.

Direct combinatorial proofs for the recurrences we found for $r = 3$ are likely to be very hard. One hopes that a uniform approach can be found to deal with all such slices of the array $a_{n,k}$ and understand the resulting chain of ideals. For the time being, we can only prove (1) and (2). The rest of the paper is dedicated to this proof.

3. Proof of Theorem 1

Most of the standard notation and terminology of the representation theory of symmetric groups can be found in [6] or [7].

By the Robinson–Schensted correspondence,

$$a_{n,k} = \sum_{\substack{\lambda\vdash n\\ \lambda_1=k}} (f^{\lambda})^2 = \sum_{\substack{\nu\vdash n-k\\ \nu_1\leq k}} (f^{(k,\nu)})^2. \tag{4}$$

We first define some auxiliary objects. Let x_λ be formal basis vectors indexed by integer partitions, and define the real vector space

$$P_m := \operatorname{span}\{x_\lambda \mid \lambda \vdash m\}$$

for every $m \geq 0$, equipped with the standard inner product.

For every positive integer s, let D_s be the rim-hook removal operator

$$D_s x_\lambda := \sum_{\substack{\mu\vdash|\lambda|-s\\ \lambda\setminus\mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda\setminus\mu)} x_\mu,$$

and let its adjoint $U_s = D_s^*$ be the rim-hook addition operator

$$U_s x_\mu := \sum_{\substack{\lambda\vdash|\mu|+s\\ \lambda\setminus\mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda\setminus\mu)} x_\lambda.$$

We also define the "regular representation" element

$$R_N := \sum_{\lambda \vdash N} f^{\lambda} x_{\lambda} \in P_N.$$

The standard character restriction and induction relationship on symmetric groups can be represented succinctly by

$$D_1 R_{n+1} = (n+1) R_n, \qquad U_1 R_n = R_{n+1}.$$

For $l, m \geq 0$, let

$$W_{l,m} := \sum_{\substack{\lambda \vdash m \\ \lambda_1 \leq l}} f^{(l,\lambda)} x_{\lambda} \in P_m.$$

Then (4) says that

$$a_{n,k} = \|W_{k,n-k}\|^2. \tag{5}$$

Lemma 1. *For every $V \in P_m$,*

$$\sum_{s=1}^{m} \|D_s V\|^2 = m \|V\|^2. \tag{6}$$

Proof. We write

$$V = \sum_{\lambda \vdash m} v_{\lambda} x_{\lambda}.$$

Standard character orthogonality implies

$$\|V\|^2 = \frac{1}{m!} \sum_{\sigma \in S_m} \left(\sum_{\lambda \vdash m} v_{\lambda} \chi_{\lambda}(\sigma) \right)^2. \tag{7}$$

On the other hand, the Murnaghan–Nakayama rule gives

$$D_s V = \sum_{\lambda \vdash m} v_{\lambda} \sum_{\substack{\mu \vdash m-s \\ \lambda \setminus \mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda \setminus \mu)} x_{\mu},$$

and therefore

$$\begin{aligned} \|D_s V\|^2 &= \frac{1}{(m-s)!} \sum_{\tau \in S_{m-s}} \left(\sum_{\substack{\lambda \vdash m \\ \mu \vdash m-s \\ \lambda \setminus \mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda \setminus \mu)} v_{\lambda} \chi_{\mu}(\tau) \right)^2 \\ &= \frac{1}{(m-s)!} \sum_{\tau \in S_{m-s}} \left(\sum_{\lambda \vdash m} v_{\lambda} \chi_{\lambda}(c_s \tau) \right)^2, \end{aligned} \tag{8}$$

where c_s is an s-cycle disjoint from τ.

Combining (7) and (8), we see that (6) is equivalent to

$$\sum_{\sigma \in S_m} \left(\sum_{\lambda \vdash m} v_{\lambda} \chi_{\lambda}(\sigma) \right)^2 = \sum_{s=1}^{m} \frac{(m-1)!}{(m-s)!} \sum_{\tau \in S_{m-s}} \left(\sum_{\lambda \vdash m} v_{\lambda} \chi_{\lambda}(c_s \tau) \right)^2.$$

This last identity can be proved bijectively. Let S_m act on $\{1, 2, \ldots, m\}$. There are exactly

$$\binom{m-1}{s-1} (s-1)! = \frac{(m-1)!}{(m-s)!}$$

possible s-cycles containing m, and every permutation is uniquely the product of its cycle containing m and a permutation on the remaining points. □

Lemma 2. *For $n \geq s \geq 2$, one has $D_s R_n = 0$.*

Proof. Let $\varphi : S_s \times S_{n-s} \to S_n$ be the natural embedding, and let π be an s-cycle in S_s. Since the character of the regular representation is zero on every non-identity element, for every $\sigma \in S_{n-s}$ we have

$$\begin{aligned} 0 &= \sum_{\lambda \vdash n} f^\lambda \chi_\lambda(\varphi(\pi,\sigma)) \\ &= \sum_{\lambda \vdash n} f^\lambda \sum_{\substack{\mu \vdash n-s \\ \lambda\backslash\mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda\backslash\mu)} \chi_\mu(\sigma) \\ &= \sum_{\mu \vdash n-s} \chi_\mu(\sigma) \sum_{\substack{\lambda \vdash n \\ \lambda\backslash\mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda\backslash\mu)} f^\lambda. \end{aligned}$$

Since the irreducible characters of S_{n-s} are linearly independent as class functions, it follows that

$$\sum_{\substack{\lambda \vdash n \\ \lambda\backslash\mu \text{ is a rim hook}}} (-1)^{\mathrm{ht}(\lambda\backslash\mu)} f^\lambda = 0, \qquad \mu \vdash n-s.$$

This is exactly the x_μ coefficient of $D_s R_n$. □

In the long-first-row regime, Lemma 2 has a particularly simple consequence.

Lemma 3. *If $m \le k+1$ and $2 \le s \le m$, then*

$$D_s W_{k,m} = -W_{k+s,m-s}. \tag{9}$$

Proof. Fix $\lambda \vdash m-s$. Since $s \ge 2$, we have $|\lambda| \le k-1$, so (k,λ) is a partition. In the coefficient of $x_{(k,\lambda)}$ in $D_s R_{k+m}$, an added rim hook is either the horizontal hook producing $(k+s,\lambda)$ or lies entirely below the first row and produces some (k,μ) occurring in $W_{k,m}$. A hook meeting both parts is impossible: once at least one new cell is put in the first row, the lower part has at most $m-1 \le k$ cells and therefore cannot reach column $k+1$ to connect to it. Consequently, the coefficient $[x_{(k,\lambda)}]D_s R_{k+m}$ (that vanishes because of Lemma 2) is $f^{(k+s,\lambda)}$ plus the coefficient of x_λ in $D_s W_{k,m}$, proving (9). □

We shall also need the ordinary one-box branching relations in the same notation.

Lemma 4. *For $l \ge 1$,*

$$D_1 W_{l,m} = (l+m)W_{l,m-1} - W_{l+1,m-1}, \qquad 1 \le m \le l+1, \tag{10}$$

$$U_1 W_{l,m} = W_{l,m+1} - W_{l-1,m+1}, \qquad 0 \le m < l. \tag{11}$$

Proof. Fix $\nu \vdash m-1$. Under the assumption $m \le l+1$, we have $\nu_1 \le l$, and the coefficient of x_ν in $D_1 W_{l,m}$ is the sum of $f^{(l,\lambda)}$ over all ways of adding one box below the first row of (l,ν). The ordinary branching rule gives

$$(l+m) f^{(l,\nu)} = f^{(l+1,\nu)} + \sum_{\substack{\lambda \vdash m,\ \lambda_1 \le l \\ \lambda\backslash\nu \text{ is one box}}} f^{(l,\lambda)},$$

which proves (10) coefficientwise.

For (11), fix $\mu \vdash m+1$. Since $m < l$, one has $\mu_1 \le l$. The coefficient of x_μ in $U_1 W_{l,m}$ is the sum of $f^{(l,\lambda)}$ over the corners of μ whose removal gives λ. These are exactly the removals below the first row of the shape (l,μ). If $\mu_1 < l$, the remaining first-row removal contributes $f^{(l-1,\mu)}$; if $\mu_1 = l$, that box is not a corner and the coefficient of x_μ in $W_{l-1,m+1}$ is zero. In either case the branching rule gives precisely the coefficient of $W_{l,m+1} - W_{l-1,m+1}$. □

Proof of (1). Put $m = n-k$. First suppose that $1 \le m < k$. From (11), adjointness, and then (10),

$$\begin{aligned} &\|W_{k,m+1}\|^2 - \|W_{k-1,m+1}\|^2 \\ &\quad = \langle U_1 W_{k,m}, W_{k,m+1} + W_{k-1,m+1}\rangle \\ &\quad = \langle W_{k,m}, D_1 W_{k,m+1} + D_1 W_{k-1,m+1}\rangle \\ &\quad = \big\langle W_{k,m}, nW_{k,m} + nW_{k-1,m} - W_{k+1,m}\big\rangle. \end{aligned}$$

Similarly,

$$\begin{aligned} &n^2\|W_{k,m-1}\|^2 - \|W_{k+1,m-1}\|^2 \\ &\quad = \langle D_1 W_{k,m}, nW_{k,m-1} + W_{k+1,m-1}\rangle \\ &\quad = \big\langle W_{k,m}, U_1\big(nW_{k,m-1} + W_{k+1,m-1}\big)\big\rangle \\ &\quad = \big\langle W_{k,m}, (n-1)W_{k,m} - nW_{k-1,m} + W_{k+1,m}\big\rangle. \end{aligned}$$

Adding the two identities cancels the mixed inner products and gives

$$\|W_{k,m+1}\|^2 + n^2\|W_{k,m-1}\|^2 = \|W_{k+1,m-1}\|^2 + (2n-1)\|W_{k,m}\|^2 + \|W_{k-1,m+1}\|^2.$$

By (5), this is (1).

The remaining case $m = 0$, or $k = n$, is immediate from $a_{n,n} = 1$, $a_{n+1,n} = n^2$, and $a_{n,n-1} = (n-1)^2$. □

Proof of (2). Again put $m = n - k$. The case $m = 0$ is immediate from $a_{n,n} = a_{n+1,n+1} = 1$ and $a_{n+1,n} = n^2$, so assume $1 \le m \le k$.

Apply Lemma 1 and Lemma 3 first to $W_{k,m+1}$ and then to $W_{k+1,m}$. We obtain

$$(m+1)\|W_{k,m+1}\|^2 = \|D_1 W_{k,m+1}\|^2 + \sum_{s=2}^{m+1} \|W_{k+s,m+1-s}\|^2,$$

$$m\|W_{k+1,m}\|^2 = \|D_1 W_{k+1,m}\|^2 + \sum_{s=3}^{m+1} \|W_{k+s,m+1-s}\|^2.$$

After subtraction,

$$(m+1)\|W_{k,m+1}\|^2 - m\|W_{k+1,m}\|^2 = \|D_1 W_{k,m+1}\|^2 - \|D_1 W_{k+1,m}\|^2 + \|W_{k+2,m-1}\|^2. \tag{12}$$

Using (10) twice, the right-hand side of (12) is

$$(n+1)^2\big(\|W_{k,m}\|^2 - \|W_{k+1,m-1}\|^2\big) + \|W_{k+1,m}\|^2 + 2(n+1)\big(\langle W_{k+1,m-1}, W_{k+2,m-1}\rangle - \langle W_{k,m}, W_{k+1,m}\rangle\big).$$

The remaining pair of mixed products is eliminated by adjointness:

$$\begin{aligned}(n+1)\|W_{k+1,m-1}\|^2 - \langle W_{k+1,m-1}, W_{k+2,m-1}\rangle &= \langle W_{k+1,m-1}, D_1 W_{k+1,m}\rangle \\ &= \langle U_1 W_{k+1,m-1}, W_{k+1,m}\rangle \\ &= \|W_{k+1,m}\|^2 - \langle W_{k,m}, W_{k+1,m}\rangle.\end{aligned}$$

Consequently,

$$(m+1)\|W_{k,m+1}\|^2 - m\|W_{k+1,m}\|^2 = (n+1)^2\big(\|W_{k,m}\|^2 + \|W_{k+1,m-1}\|^2\big) - (2n+1)\|W_{k+1,m}\|^2.$$

Rearranging and using $2n + 1 - m = n + k + 1$ gives

$$(n+1)^2\big(\|W_{k,m}\|^2 + \|W_{k+1,m-1}\|^2\big) = (m+1)\|W_{k,m+1}\|^2 + (n+k+1)\|W_{k+1,m}\|^2.$$

By (5), this is exactly (2). □

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