# A PROOF OF CONJECTURE 16

Manuel Kauers · Institute for Algebra · JKU

Joint work with Frederic Chyzak and Hui Huang

3, 5, 7, 9, 11, ??

$$3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

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$$(2n+3)f(n+1) - (2n+5)f(n) = 0$$

$$3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

$$(2n+3)f(n+1) - (2n+5)f(n) = 0$$

$2, 2, 3, 6, 15, 43, ??$

$$3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

$$(2n+3)f(n+1) - (2n+5)f(n) = 0$$

$$2, 2, 3, 6, 15, 43, 133, 430, 1431, 4863, 16797, 58787, ??$$

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$$2, 2, 3, 6, 15, 43, 133, 430, 1431, 4863, 16797, 58787, ??$$

$$n(n+3)f(n+2) - (5n^2+9n+2)f(n+1) + 2(n+1)(2n+1)f(n) = 0$$

$$3, 5, 7, 9, 11, 13, 15, 17, 19, \ldots$$

$$(2n+3)f(n+1) - (2n+5)f(n) = 0$$

$$2, 2, 3, 6, 15, 43, 133, 430, 1431, 4863, 16797, 58787, 208013, 742901, \ldots$$

$$n(n+3)f(n+2) - (5n^2+9n+2)f(n+1) + 2(n+1)(2n+1)f(n) = 0$$

Input:

- A finite sequence of numbers, $a_0, a_1, \ldots, a_N$
- two numbers, $r, d$

Input:

- A finite sequence of numbers, $a_0, a_1, \ldots, a_N$
- two numbers, $r, d$

Output:

- A linear recurrence equation of order $\leq r$ with polynomial coefficients of degree $\leq d$ that matches the given sequence.

$$2, 2, 3, 6, 15, 43, 133, 430, 1431, 4863, 16797, 58787, 208013, 742901, \ldots$$

To catch a recurrence with $r = d = 2$, the classical technique needs at least $12$ terms.

$$\overbrace{2, 2, 3, 6, 15, 43, 133, 430, 1431, 4863, 16797, 58787,}\ 208013, 742901, \ldots$$

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$$\overbrace{\underbrace{2, 2, 3, 6, 15, 43, 133}, 430, 1431, 4863, 16797, 58787}, 208013, 742901, \ldots$$

A recent alternative technique succeeds with only 7 terms.

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# Guessing with Little Data*

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## ABSTRACT
Reconstructing a hypothetical recurrence equation from the first terms of an infinite sequence is a classical and well-known technique in experimental mathematics. We propose a variation of this technique which can succeed with fewer input terms.

## CCS CONCEPTS
• **Computing methodologies** → **Algebraic algorithms**.

## KEYWORDS
Experimental Mathematics, D-finite Functions, Lattice Reduction, Integer Sequences

**ACM Reference Format:**
Manuel Kauers and Christoph Koutschan. 2022. Guessing with Little Data. In *Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation (ISSAC '22), July 4–7, 2022, Villeneuve-d'Ascq, France.* ACM, New York, NY, USA, 8 pages. https://doi.org/10.1145/3476446.3535486

sequence, we can instantiate the ansatz for $n = 0, \ldots, N - r$ and get $N - r + 1$ linear constraints on the $(r + 1)(d + 1)$ unknown coefficients $c_{i,j}$. If $r$ and $d$ are chosen such that $(r + 1)(d + 2) \leq N + 2$, this homogeneous linear system is not expected to have a (nontrivial) solution. If it does have a solution, this is interpreted as evidence in favor of the correctness of the corresponding equation. The more the number of equations exceeds the number of variables, the stronger is the evidence that the solution is not just noise but means something.

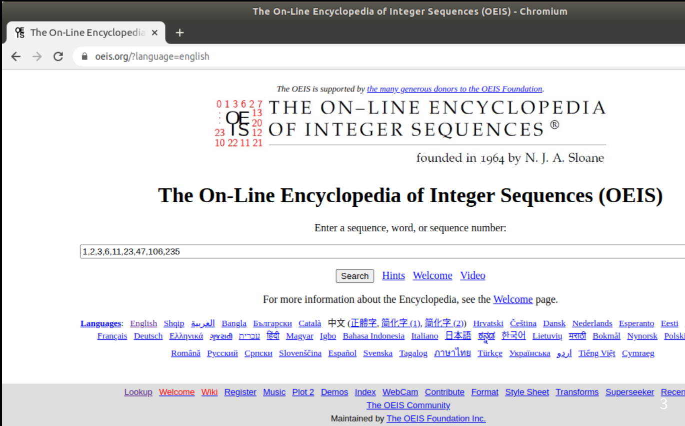
The linear systems arising from guessing problems can be solved efficiently by Hermite-Padé approximation [1, 8, 13, 15]. Modern implementations have no trouble handling examples where $N$ is in the range of 10000. There are also variants for the multivariate setting [2–4] as well as experiments with approaches that use machine learning instead of linear algebra [12].

Although it rarely happens in practice, a guessed equation may be incorrect. It is therefore important to be able to assess the quality of a guess. The amount of overdetermination of the linear system

To catch a recurrence with $r = d = 2$, the classical technique needs at least 12 terms.

$$\overbrace{\underbrace{2, 2, 3, 6, 15, 43, 133}, 430, 1431, 4863, 16797, 58787}, 208013, 742901, \ldots$$

A recent alternative technique succeeds with only 7 terms.



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Journal of Integer Sequences, Vol. 26 (2023),
Article 23.4.5

**Some D-Finite and Some Possibly D-Finite Sequences in the OEIS**

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| Entry | First terms | $N_{oeis}$ | $N_{linalg}$ | $N_{LLL}$ | $r$ | $d$ |
|---|---|---|---|---|---|---|
| A177317 | 1, 2, 48, 2288, 135040 | 29 | 60 | 22 | 3 | 14 |
| A199250 | 1, 1, 14, 21, 424, 571 | 56 | 98 | 56 | 8 | 18 |
| A250556 | 8, 60, 302, 1516, 7126 | 47 | 58 | 47 | 9 | 8 |
| A265234 | 1, 43, 2592, 184740 | 31 | 56 | 27 | 6 | 6 |
| A172572 | 90, 67950, 90291600 | 33 | 44 | 17 | 3 | 9 |
| A172671 | 90, 202410, 747558000 | 33 | 75 | 25 | 4 | 13 |
| A188818 | 2, 9, 48, 256, 1360 | 32 | 55 | 26 | 5 | 10 |
| A306322 | 1, 0, 0, 25, 386, 4657 | 41 | 63 | 30 | 4 | 14 |
| A195806 | 16, 105, 496, 1759, 5052 | 32 | 41 | 30 | 4 | 10 |
| A216940 | 260, 27768, 1664244 | 37 | 44 | 29 | 1 | 23 |
| A194478 | 0, 0, 0, 1, 337, 8733 | 32 | 35 | 19 | 2 | 12 |
| A215570 | 1, 35, 18720, 19369350 | 48 | 68 | 27 | 3 | 15 |
| A339987 | 1, 4, 90, 8400, 1426950 | 40 | 70 | 24 | 5 | 10 |
| A269021 | 1, 2, 23, 588, 24553 | 42 | 108 | 28 | 4 | 21 |
| A181198 | 1, 1, 8, 169, 6392 | 27 | 33 | 14 | 2 | 9 |
| A181199 | 1, 1, 16, 985, 141696 | 26 | 103 | 34 | 3 | 24 |
| A181280 | 0, 0, 0, 58, 1629, 28924 | 27 | 32 | 26 | 10 | 1 |
| A253217 | 0, 0, 1, 19, 268, 3568 | 37 | 53 | 27 | 5 | 9 |
| A098926 | 0, 2, 12, 90, 556, 5242 | 34 | 55 | 26 | 8 | 7 |
| A164735 | 0, 0, 0, 0, 0, 0, 0, 1, 0, 4 | 70 | 80 | 66 | 15 | 4 |

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|---|---|---|---|---|---|---|---|
| A177317 | 1, 2, 48, 2288, 135040 | 29 | 60 | 22 | 3 | 14 | ✓ |
| A199250 | 1, 1, 14, 21, 424, 571 | 56 | 98 | 56 | 8 | 18 | ✓ |
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| A172671 | 90, 202410, 747558000 | 33 | 75 | 25 | 4 | 13 | ✓ |
| A188818 | 2, 9, 48, 256, 1360 | 32 | 55 | 26 | 5 | 10 | ✓ |
| A306322 | 1, 0, 0, 25, 386, 4657 | 41 | 63 | 30 | 4 | 14 | ✓ |
| A195806 | 16, 105, 496, 1759, 5052 | 32 | 41 | 30 | 4 | 10 | ✓ |
| A216940 | 260, 27768, 1664244 | 37 | 44 | 29 | 1 | 23 | ✓ |
| A194478 | 0, 0, 0, 1, 337, 8733 | 32 | 35 | 19 | 2 | 12 | ✓ |
| A215570 | 1, 35, 18720, 19369350 | 48 | 68 | 27 | 3 | 15 | ? |
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| A181198 | 1, 1, 8, 169, 6392 | 27 | 33 | 14 | 2 | 9 | ? |
| A181199 | 1, 1, 16, 985, 141696 | 26 | 103 | 34 | 3 | 24 | ? |
| A181280 | 0, 0, 0, 58, 1629, 28924 | 27 | 32 | 26 | 10 | 1 | ✓ |
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| Entry | First terms | $N_{oeis}$ | $N_{linalg}$ | $N_{LLL}$ | r | d | |
|---|---|---|---|---|---|---|---|
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| A194478 | 0, 0, 0, 1, 337, 8733 | 32 | 35 | 19 | 2 | 12 | ✓ |
| A215570 | 1, 35, 18720, 19369350 | 48 | 68 | 27 | 3 | 15 | ? |
| A339987 | 1, 4, 90, 8400, 1426950 | 40 | 70 | 24 | 5 | 10 | ✓ |
| A269021 | 1, 2, 23, 588, 24553 | 42 | 108 | 28 | 4 | 21 | ? |
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| A181280 | 0, 0, 0, 58, 1629, 28924 | 27 | 32 | 26 | 10 | 1 | ✓ |
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| A164735 | 0, 0, 0, 0, 0, 0, 0, 1, 0, 4 | 70 | 80 | 66 | 15 | 4 | ? |

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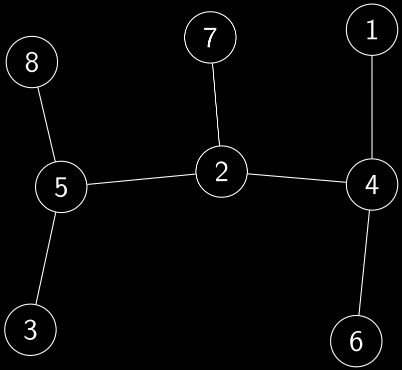
$$= \boxed{\text{Number of labeled graphs with } 2n \text{ vertices and } 2n-1 \text{ edges and all vertex degrees in } \{1,3\}.}$$


1
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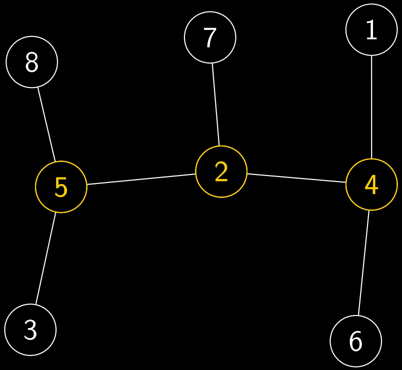

8
7
1
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2
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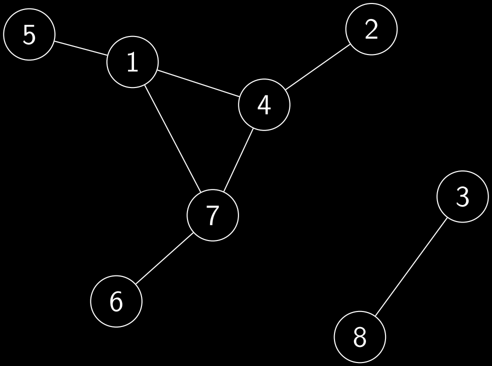
3
6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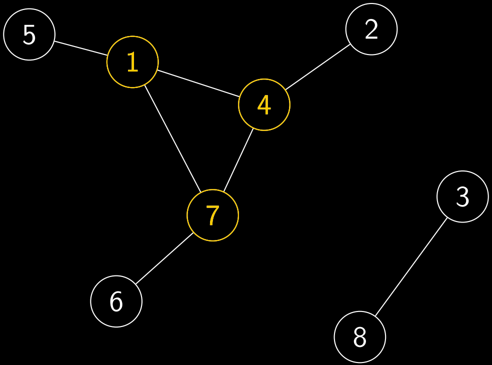
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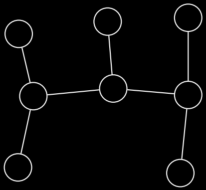
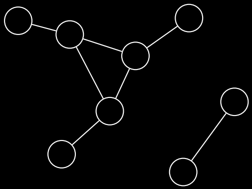


5400 labelings



3360 labelings

$$\Rightarrow \quad a_4 = 5400 + 3360 = \underline{8400}$$

**Conjecture 16:** If $(a_n)$ denotes the sequence A339987, then

$$
\begin{aligned}
&32(328n^3 + 3300n^2 + 10844n + 11589) \\
&\qquad \times (n+1)(n+2)(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)\, a_n \\
&- 8(2624n^4 + 30664n^3 + 129460n^2 + 232328n + 148119) \\
&\qquad \times (n+2)(2n+3)(2n+5)(2n+7)(2n+9)\, a_{n+1} \\
&- 16(2952n^5 + 40852n^4 + 219308n^3 + 569267n^2 + 712135n + 341634) \\
&\qquad \times (n+3)(2n+5)(2n+7)(2n+9)\, a_{n+2} \\
&+ 8(3936n^5 + 55672n^4 + 306380n^3 + 818282n^2 + 1057879n + 527520) \\
&\qquad \times (n+4)(2n+7)(2n+9)\, a_{n+3} \\
&- 2(2624n^5 + 42472n^4 + 264028n^3 + 786236n^2 + 1117119n + 601452) \\
&\qquad \times (n+5)(2n+9)\, a_{n+4} \\
&+ 3(328n^3 + 2316n^2 + 5228n + 3717)(n+4)(n+6)\, a_{n+5} = 0.
\end{aligned}
$$

$$x_1 + x_2 + x_3$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots$$

$$p_1 := x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots$$

$$p_1 := x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots$$
$$p_2 := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + \cdots$$

$$
\begin{aligned}
p_1 &:= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots \\
p_2 &:= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + \cdots \\
p_3 &:= x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + \cdots \\
p_4 &:= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + \cdots \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
p_1 &:= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots \\
p_2 &:= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + \cdots \\
p_3 &:= x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + \cdots \\
p_4 &:= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + \cdots \\
&\ \ \vdots
\end{aligned}
$$

Fact: $\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} = \mathbb{C}[[p_1, p_2, p_3, \ldots]]$

$$
\begin{aligned}
p_1 &:= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots \\
p_2 &:= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + \cdots \\
p_3 &:= x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 + x_7^3 + \cdots \\
p_4 &:= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + \cdots \\
&\;\vdots
\end{aligned}
$$

Fact: $\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} = \mathbb{C}[[p_1, p_2, p_3, \ldots]]$
$= \mathbb{C}[[h_1, h_2, h_3, \ldots]]$

$$h_1 := x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots$$

Fact: $$\begin{aligned}\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} &= \mathbb{C}[[p_1, p_2, p_3, \ldots]] \\ &= \mathbb{C}[[h_1, h_2, h_3, \ldots]]\end{aligned}$$

$$h_1 := x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots$$
$$h_2 := x_1\, x_2 + x_1\, x_3 + x_2\, x_3 + \cdots + x_{37}\, x_{57} + \cdots$$

Fact: $\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} = \mathbb{C}[[p_1, p_2, p_3, \ldots]]$
$\qquad\qquad\qquad\qquad\qquad = \mathbb{C}[[h_1, h_2, h_3, \ldots]]$

$$
\begin{aligned}
h_1 &:= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \cdots \\
h_2 &:= x_1\,x_2 + x_1\,x_3 + x_2\,x_3 + \cdots + x_{37}\,x_{57} + \cdots \\
h_3 &:= x_1\,x_2\,x_3 + x_1\,x_2\,x_4 + \cdots + x_{74}\,x_{93}\,x_{136} + \cdots \\
h_4 &:= x_1\,x_2\,x_3\,x_4 + x_1\,x_2\,x_3\,x_5 + \cdots + x_{19}\,x_{63}\,x_{87}\,x_{174} + \cdots \\
&\ \ \vdots
\end{aligned}
$$

Fact: $\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} = \mathbb{C}[[p_1, p_2, p_3, \ldots]]$
$\qquad\qquad\qquad\qquad\qquad = \mathbb{C}[[h_1, h_2, h_3, \ldots]]$

$$
\begin{aligned}
h_1 &= p_1 \\
h_2 &= \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2 \\
h_3 &= \tfrac{1}{6}p_1^3 + \tfrac{1}{2}p_1\,p_2 + \tfrac{1}{3}p_3 \\
h_4 &= \tfrac{1}{24}p_1^4 + \tfrac{1}{4}p_1^2p_2 + \tfrac{1}{3}p_1\,p_3 + \tfrac{1}{8}p_2^2 + \tfrac{1}{4}p_4 \\
&\ \ \vdots
\end{aligned}
$$

**Fact:** $$\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} = \mathbb{C}[[p_1, p_2, p_3, \ldots]] = \mathbb{C}[[h_1, h_2, h_3, \ldots]]$$

$$
\begin{aligned}
p_1 &= h_1 \\
p_2 &= h_1^2 + 2h_2 \\
p_3 &= h_1^3 - 3h_1h_2 + 3h_3 \\
p_4 &= -h_1^4 + 4h_1^2h_2 - 2h_2^2 - 4h_1h_3 + 4h_4 \\
&\ \vdots
\end{aligned}
$$

Fact: $\mathbb{C}[[x_1, x_2, x_3, \ldots]]_{\mathsf{sym}} = \mathbb{C}[[p_1, p_2, p_3, \ldots]]$
$= \mathbb{C}[[h_1, h_2, h_3, \ldots]]$

$$\prod_{i<j} (1 + x_i x_j)$$

$$[x_1^{\lambda_1} \cdots x_n^{\lambda_n}] \prod_{i<j} \left(1 + x_i x_j\right)$$

$$[x_1^{\lambda_1} \cdots x_n^{\lambda_n}] \prod_{i<j} \left(1 + x_i x_j\right)$$

$$= \boxed{\text{Number of graphs with vertices } 1, \ldots, n \text{ and } \lambda_i \text{ the degree of vertex } i \ (i = 1, \ldots, n).}$$

no edge

edge

$$[x_1^{\lambda_1} \cdots x_n^{\lambda_n}] \prod_{i<j} \big(1 + x_i x_j\big)$$

$$= \boxed{\text{Number of graphs with vertices } 1, \ldots, n \text{ and } \lambda_i \text{ the degree of vertex } i \ (i = 1, \ldots, n).}$$

no edge

edge

$$[x_1^3 \cdots x_n^3] \prod_{i<j} \left(1 + x_i x_j\right)$$

$$= \boxed{\text{Number of graphs with vertices } 1, \ldots, n \text{ and all vertex degrees equal to } 3.}$$

no edge

edge

$$\sum_{(\lambda_1,\dots,\lambda_n)\in\{1,3\}^n} [x_1^{\lambda_1}\cdots x_n^{\lambda_n}] \prod_{i<j}(1 + x_i x_j)$$

$$= \boxed{\text{Number of graphs with vertices } 1,\dots,n \text{ and all vertex degrees in } \{1,3\}.}$$

For $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$, let $m_{\lambda_1,\ldots,\lambda_n}$ be the sum of all distinct monomials $x_{i_1}^{\lambda_1} \cdots x_{i_n}^{\lambda_n}$ where $i_1, \ldots, i_n$ are pairwise distinct.

For $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$, let $m_{\lambda_1,\ldots,\lambda_n}$ be the sum of all distinct monomials $x_{i_1}^{\lambda_1} \cdots x_{i_n}^{\lambda_n}$ where $i_1, \ldots, i_n$ are pairwise distinct.

There is a scalar product

$$\langle \cdot | \cdot \rangle \colon \mathbb{C}[[x_1, x_2, \ldots]]_{\mathsf{sym}} \times \mathbb{C}[[x_1, x_2, \ldots]]_{\mathsf{sym}} \to \mathbb{C}$$

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with

$$\langle m_{\lambda_1,\dots,\lambda_n} \mid h_{\mu_1} \cdots h_{\mu_k} \rangle := \begin{cases} 1 & \text{if } (\lambda_1, \dots, \lambda_n) = (\mu_1, \dots, \mu_k) \\ 0 & \text{else} \end{cases}$$

$$\sum_{(\lambda_1,\ldots,\lambda_n)\in\{1,3\}^n} [x_1^{\lambda_1}\cdots x_n^{\lambda_n}] \prod_{i<j}(1+x_i x_j)$$

$$= \boxed{\begin{array}{l}\text{Number of labeled graphs with } n \text{ vertices} \\ \text{and all vertex degrees in } \{1,3\}\end{array}}$$

$$\langle \prod_{i<j} (1 + x_i x_j) \mid (h_1 + h_3)^n \rangle$$

$$= \boxed{\text{Number of labeled graphs with } n \text{ vertices and all vertex degrees in } \{1, 3\}}$$

$$\left\langle \prod_{i<j}(1 + x_i x_j) \;\middle|\; \sum_{n=0}^{\infty}(h_1 + h_3)^n t^n \right\rangle$$

$$= \sum_{n=0}^{\infty} \boxed{\begin{array}{l}\text{Number of labeled graphs with } n \text{ vertices} \\ \text{and all vertex degrees in } \{1, 3\}\end{array}} \, t^n$$

$$\left\langle \prod_{i<j}(1+x_ix_j) \;\middle|\; \frac{1}{1-(h_1+h_3)t} \right\rangle$$

$$= \sum_{n=0}^{\infty} \boxed{\begin{array}{l}\text{Number of labeled graphs with } n \text{ vertices} \\ \text{and all vertex degrees in } \{1,3\}\end{array}} \; t^n$$

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$$= \sum_{n=0}^{\infty} \boxed{\qquad\qquad\qquad} \, t^n q^k$$

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Fact:

$$\langle p_1^{s_1} \cdots p_\ell^{s_\ell} \mid p_1^{r_1} \cdots p_m^{r_m} \rangle$$

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Hence:

Fact:

$$\langle p_1^{s_1}\cdots p_\ell^{s_\ell} \mid p_1^{r_1}\cdots p_m^{r_m}\rangle$$
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Hence:

$$\langle f \mid g\rangle = \left[\, f \;\odot_{p_1,p_2,\ldots}\; g \;\odot_{p_1,p_2,\ldots}\; \prod_{r=0}^{\infty}\left(\sum_{n=0}^{\infty} r^n n! p_r^n\right)\,\right]_{p_1=p_2=\cdots=1}$$

$$\left\langle \prod_{i<j} (1 + q x_i x_j) \;\middle|\; \frac{1}{1 - (h_1 + h_3) t} \right\rangle$$

$$\langle \prod_{i<j} (1 + q x_i x_j) \mid \frac{1}{1 - (p_1 + \frac{1}{6} p_1^3 + \frac{1}{2} p_1 p_2 + \frac{1}{3} p_3) t} \rangle$$

$$\left\langle \exp\ \log \prod_{i<j}(1 + q x_i x_j) \mid \frac{1}{1 - (p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3)t} \right\rangle$$

$$\left\langle \exp \sum_{i<j} \log(1 + q x_i x_j) \;\middle|\; \frac{1}{1 - (p_1 + \frac{1}{6} p_1^3 + \frac{1}{2} p_1 p_2 + \frac{1}{3} p_3) t} \right\rangle$$

$$\left\langle \exp \sum_{i<j} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (q x_i x_j)^n \;\middle|\; \frac{1}{1-(p_1+\frac{1}{6}p_1^3+\frac{1}{2}p_1 p_2+\frac{1}{3}p_3)t} \right\rangle$$

$$\left\langle \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^n \sum_{i<j} (x_i x_j)^n \;\middle|\; \frac{1}{1-(p_1+\frac{1}{6}p_1^3+\frac{1}{2}p_1p_2+\frac{1}{3}p_3)t} \right\rangle$$

$$\left\langle \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^n \sum_{i<j} (x_i x_j)^n \;\middle|\; \frac{1}{1-(p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3)t} \right\rangle$$

$p_n^2$

$$\left\langle \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^n \sum_{i<j} (x_i x_j)^n \;\middle|\; \frac{1}{1-(p_1+\frac{1}{6}p_1^3+\frac{1}{2}p_1p_2+\frac{1}{3}p_3)t} \right\rangle$$

$$p_n^2 = \left(\sum_i x_i^n\right)\left(\sum_j x_j^n\right)$$

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$$p_n^2 = \left(\sum_i x_i^n\right)\left(\sum_j x_j^n\right) = \sum_{i,j} x_i^n x_j^n$$

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$$p_n^2 = \left(\sum_i x_i^n\right)\left(\sum_j x_j^n\right) = \sum_{i,j} x_i^n x_j^n = 2\sum_{i<j} x_i^n x_j^n + \sum_i x_i^{2n}$$

$$\langle \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^n \sum_{i<j} (x_i x_j)^n \mid \frac{1}{1-(p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3)t} \rangle$$

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$$\langle \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^n \frac{p_n^2 - p_{2n}}{2} \mid \frac{1}{1-(p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3)t} \rangle$$

$$\left\langle \exp \sum_{n=1}^{\infty} \frac{(-1)^n}{n} q^n \frac{p_n^2 - p_{2n}}{2} \;\middle|\; \frac{1}{1 - (p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3)t} \right\rangle$$

$$\left\langle \exp \sum_{n=1}^{3} \frac{(-1)^n}{n} q^n \frac{p_n^2 - p_{2n}}{2} \;\middle|\; \frac{1}{1 - (p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1 p_2 + \frac{1}{3}p_3)t} \right\rangle$$

$$\left\langle \exp\left(\frac{1}{2}q(p_1^2 - p_2) - \frac{1}{4}q^2p_2^2 + \frac{1}{6}q^3p_3^2\right) \,\middle|\, \frac{1}{1-(p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1p_2 + \frac{1}{3}p_3)t} \right\rangle$$

$$f(q,t) = \Bigg[\exp\left(\tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2p_2^2 + \tfrac{1}{6}q^3p_3^2\right)$$

$$\odot_{p_1,p_2,p_3} \quad \frac{1}{1-(p_1+\frac{1}{6}p_1^3+\frac{1}{2}p_1p_2+\frac{1}{3}p_3)t}$$

$$\odot_{p_1,p_2,p_3} \quad \left(\sum_{n=0}^{\infty} n!p_1^n\right)\left(\sum_{n=0}^{\infty} 2^n n!p_2^n\right)\left(\sum_{n=0}^{\infty} 3^n n!p_3^n\right)\Bigg]_{p_1=p_2=p_3=1}$$

$$f(q,t) = \Bigg[\exp\left(\tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2p_2^2 + \tfrac{1}{6}q^3p_3^2\right)$$

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This is D-finite.

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Then also $\sum_{n=0}^{\infty} a_n t^{2n} = \operatorname{diag}_q f(q,t)$ is D-finite.

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Then also $\sum_{n=0}^{\infty} a_n t^{2n} = \operatorname{diag}_q f(q,t)$ is D-finite.

So $(a_n)_{n=0}^{\infty}$ does satisfy a certain linear recurrence.

JOURNAL OF COMBINATORIAL THEORY, Series A **53**, 257–285 (1990)

# Symmetric Functions and P-Recursiveness

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*Communicated by the Managing Editors*

Received May 23, 1988

## 1. INTRODUCTION

Many enumeration problems, such as that of counting nonnegative integer matrices with given row and column sums, have solutions which can be expressed as coefficients of symmetric functions. We show here how useful formulas can be obtained from these symmetric function generating functions. In some cases, the symmetric functions yield reasonably simple explicit formulas or generating functions for the coefficients. In other cases, such as in counting several types of regular graphs, explicit formulas that are too unwieldy to be useful in computation can still be used to show that a sequence of coefficients is P-recursive; that is, it satisfies a linear homogeneous recurrence with polynomial coefficients.

**Conjecture 16:** If $(a_n)$ denotes the sequence A339987, then

$$
\begin{aligned}
&32(328n^3 + 3300n^2 + 10844n + 11589) \\
&\qquad \times (n+1)(n+2)(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)\, a_n \\
&- 8(2624n^4 + 30664n^3 + 129460n^2 + 232328n + 148119) \\
&\qquad \times (n+2)(2n+3)(2n+5)(2n+7)(2n+9)\, a_{n+1} \\
&- 16(2952n^5 + 40852n^4 + 219308n^3 + 569267n^2 + 712135n + 341634) \\
&\qquad \times (n+3)(2n+5)(2n+7)(2n+9)\, a_{n+2} \\
&+ 8(3936n^5 + 55672n^4 + 306380n^3 + 818282n^2 + 1057879n + 527520) \\
&\qquad \times (n+4)(2n+7)(2n+9)\, a_{n+3} \\
&- 2(2624n^5 + 42472n^4 + 264028n^3 + 786236n^2 + 1117119n + 601452) \\
&\qquad \times (n+5)(2n+9)\, a_{n+4} \\
&+ 3(328n^3 + 2316n^2 + 5228n + 3717)(n+4)(n+6)\, a_{n+5} = 0.
\end{aligned}
$$

$$
\begin{aligned}
f(q,t) = \Bigg[ & \exp\left(\tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2p_2^2 + \tfrac{1}{6}q^3p_3^2\right) \\
& \odot_{p_1,p_2,p_3} \quad \frac{1}{1 - (p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1p_2 + \frac{1}{3}p_3)t} \\
& \odot_{p_1,p_2,p_3} \quad \left(\sum_{n=0}^{\infty} n!p_1^n\right)\left(\sum_{n=0}^{\infty} 2^n n!p_2^n\right)\left(\sum_{n=0}^{\infty} 3^n n!p_3^n\right)\Bigg]_{p_1=p_2=p_3=1}
\end{aligned}
$$

$$f(q,t) = \Bigg[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2p_2^2 + \tfrac{1}{6}q^3p_3^2\right)^n$$

$$\odot_{p_1,p_2,p_3} \quad \frac{1}{1 - (p_1 + \frac{1}{6}p_1^3 + \frac{1}{2}p_1p_2 + \frac{1}{3}p_3)t}$$

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$$f(q,t) = \Bigg[\sum_{n=0}^{\infty} \frac{1}{n!}\left(\tfrac{1}{2}q(p_1^2 - p_2) - \tfrac{1}{4}q^2p_2^2 + \tfrac{1}{6}q^3p_3^2\right)^n$$

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$$\odot_{p_1,p_2,p_3} \quad \sum_{u,v,w=0}^{\infty} u!v!w!2^v3^w p_1^u p_2^v p_3^w \Bigg]_{p_1=p_2=p_3=1}$$

$$a_n = \sum_{u,v,w} \frac{(-1)^{u+v+w+1}}{2^{n-1-u+w} 3^{2u+w-n}} \times \frac{(2n)!(2w)!}{(n+1)!(u+w-n)!u!v!w!(2n-1-3u-2v-w)!}$$

# HolonomicFunctions

(User's Guide)


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June 17, 2013

This manual describes the functionality of the Mathematica package **HolonomicFunctions**. It is a very powerful tool for the work with special functions, it can assist in solving summation and integration problems, it can automatically prove special function identities, and much more. The package has been developed in the frame of the PhD thesis [9]. The whole theory and the algorithms are described there, and it contains also many references for further reading as well as some more advanced examples; the examples in this manual are mostly of a very simple nature in order to illustrate clearly the use of the software. **HolonomicFunctions** is freely available from the RISC combinatorics software webpage



www.risc.uni-linz.ac.at/research/combinat/software/HolonomicFunctions/

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- This spares the need to check boundary conditions.
- This computation provides a second rigorous proof.
- Meanwhile we also have a third (computer-free) proof based on a combinatorial argument.

| Entry | First terms | $N_{oeis}$ | $N_{linalg}$ | $N_{LLL}$ | r | d | |
|---|---|---|---|---|---|---|---|
| A177317 | 1, 2, 48, 2288, 135040 | 29 | 60 | 22 | 3 | 14 | ✓ |
| A199250 | 1, 1, 14, 21, 424, 571 | 56 | 98 | 56 | 8 | 18 | ✓ |
| A250556 | 8, 60, 302, 1516, 7126 | 47 | 58 | 47 | 9 | 8 | ✓ |
| A265234 | 1, 43, 2592, 184740 | 31 | 56 | 27 | 6 | 6 | ✓ |
| A172572 | 90, 67950, 90291600 | 33 | 44 | 17 | 3 | 9 | ✓ |
| A172671 | 90, 202410, 747558000 | 33 | 75 | 25 | 4 | 13 | ✓ |
| A188818 | 2, 9, 48, 256, 1360 | 32 | 55 | 26 | 5 | 10 | ✓ |
| A306322 | 1, 0, 0, 25, 386, 4657 | 41 | 63 | 30 | 4 | 14 | ✓ |
| A195806 | 16, 105, 496, 1759, 5052 | 32 | 41 | 30 | 4 | 10 | ✓ |
| A216940 | 260, 27768, 1664244 | 37 | 44 | 29 | 1 | 23 | ✓ |
| A194478 | 0, 0, 0, 1, 337, 8733 | 32 | 35 | 19 | 2 | 12 | ✓ |
| A215570 | 1, 35, 18720, 19369350 | 48 | 68 | 27 | 3 | 15 | ? |
| A339987 | 1, 4, 90, 8400, 1426950 | 40 | 70 | 24 | 5 | 10 | ✓ |
| A269021 | 1, 2, 23, 588, 24553 | 42 | 108 | 28 | 4 | 21 | ? |
| A181198 | 1, 1, 8, 169, 6392 | 27 | 33 | 14 | 2 | 9 | ? |
| A181199 | 1, 1, 16, 985, 141696 | 26 | 103 | 34 | 3 | 24 | ? |
| A181280 | 0, 0, 0, 58, 1629, 28924 | 27 | 32 | 26 | 10 | 1 | ✓ |
| A253217 | 0, 0, 1, 19, 268, 3568 | 37 | 53 | 27 | 5 | 9 | ✓ |
| A098926 | 0, 2, 12, 90, 556, 5242 | 34 | 55 | 26 | 8 | 7 | ✓ |
| A164735 | 0, 0, 0, 0, 0, 0, 0, 1, 0, 4 | 70 | 80 | 66 | 15 | 4 | ? |

| Entry | First terms | $N_{\text{oeis}}$ | $N_{\text{linalg}}$ | $N_{\text{LLL}}$ | $r$ | $d$ | |
|---|---|---|---|---|---|---|---|
| A177317 | 1, 2, 48, 2288, 135040 | 29 | 60 | 22 | 3 | 14 | ✓ |
| A199250 | 1, 1, 14, 21, 424, 571 | 56 | 98 | 56 | 8 | 18 | ✓ |
| A250556 | 8, 60, 302, 1516, 7126 | 47 | 58 | 47 | 9 | 8 | ✓ |
| A265234 | 1, 43, 2592, 184740 | 31 | 56 | 27 | 6 | 6 | ✓ |
| A172572 | 90, 67950, 90291600 | 33 | 44 | 17 | 3 | 9 | ✓ |
| A172671 | 90, 202410, 747558000 | 33 | 75 | 25 | 4 | 13 | ✓ |
| A188818 | 2, 9, 48, 256, 1360 | 32 | 55 | 26 | 5 | 10 | ✓ |
| A306322 | 1, 0, 0, 25, 386, 4657 | 41 | 63 | 30 | 4 | 14 | ✓ |
| A195806 | 16, 105, 496, 1759, 5052 | 32 | 41 | 30 | 4 | 10 | ✓ |
| A216940 | 260, 27768, 1664244 | 37 | 44 | 29 | 1 | 23 | ✓ |
| A194478 | 0, 0, 0, 1, 337, 8733 | 32 | 35 | 19 | 2 | 12 | ✓ |
| A215570 | 1, 35, 18720, 19369350 | 48 | 68 | 27 | 3 | 15 | ? |
| A339987 | 1, 4, 90, 8400, 1426950 | 40 | 70 | 24 | 5 | 10 | ✓ |
| A269021 | 1, 2, 23, 588, 24553 | 42 | 108 | 28 | 4 | 21 | ? |
| A181198 | 1, 1, 8, 169, 6392 | 27 | 33 | 14 | 2 | 9 | ? |
| A181199 | 1, 1, 16, 985, 141696 | 26 | 103 | 34 | 3 | 24 | ? |
| A181280 | 0, 0, 0, 58, 1629, 28924 | 27 | 32 | 26 | 10 | 1 | ✓ |
| A253217 | 0, 0, 1, 19, 268, 3568 | 37 | 53 | 27 | 5 | 9 | ✓ |
| A098926 | 0, 2, 12, 90, 556, 5242 | 34 | 55 | 26 | 8 | 7 | ✓ |
| A164735 | 0, 0, 0, 0, 0, 0, 0, 1, 0, 4 | 70 | 80 | 66 | 15 | 4 | ? |

3 4 2 7 9 1 5 6 8

3 4 2 7 9 1 5 6 8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 2 | 7 | 9 | 1 | 5 | 6 | 8 |
| | | 1 | 3 | | | 2 | | 4 |

3 4 2 7 9 1 5 6 8

1 3 2 4

How many permutations contain the pattern 1324?

3 4 2 7 9 1 5 6 8

1 3 2 4

How many permutations contain the pattern 1324?

Let $a_n$ be the number of such permutations in $S_n$

3 4 2 7 9 1 5 6 8

1 3 2 4

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Let $a_n$ be the number of such permutations in $S_n$

Famous open problem: is $a_n$ D-finite?

3 4 2 7 9 1 5 6 8

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 2 | 7 | 9 | 1 | 5 | 6 | 8 |
| 1 | 2 | | 3 | | | | | 4 |

3 4 2 7 9 1 5 6 8

1 2 3 4

How many permutations contain the pattern 1234?

3 4 2 7 9 1 5 6 8

1 2 3 4

How many permutations contain the pattern 1234?

Let $a_n$ be the number of such permutations in $S_n$

3 4 2 7 9 1 5 6 8

1 2 3 4

How many permutations contain the pattern 1234?

Let $a_n$ be the number of such permutations in $S_n$

Well-known fact: $a_n$ is D-finite.

3 4 2 7 9 1 5 6 8

1 2 3 4

How many permutations contain the pattern $12\ldots k$?

Let $a_n$ be the number of such permutations in $S_n$

Well-known fact: $a_n$ is D-finite *for every fixed* $k$.

A269021: $a_n =$ number of permutations of length $2n$, containing the pattern $12 \ldots n$.

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**Conjecture:** This sequence is D-finite.

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**Conjecture:** This sequence is D-finite.

$$-64(n+1)^2(n+2)^2(n+3)(2n+1)^2(2n+3)^2(2n+5)^2(64n^{10}+1968n^9+26156n^8+198469n^7+952323n^6+3012795n^5+6333869n^4+8663374n^3+7264534n^2+3266000n+549760)a_n+16(n+2)^2(n+3)(2n+3)^2(2n+5)^2(64n^{13}+2672n^{12}+49788n^{11}+545913n^{10}+3917758n^9+19359535n^8+67385886n^7+165789363n^6+284054698n^5+325846005n^4+229526554n^3+78563984n^2-487964n-5543040)a_{n+1}-4(n+3)(2n+5)^2(512n^{15}+21568n^{14}+419248n^{13}+4969164n^{12}+39928763n^{11}+228837227n^{10}+959068672n^9+2966908118n^8+6753094929n^7+11118771121n^6+12741784568n^5+9313604242n^4+3271711596n^3-562569136n^2-946158512n-250467360)a_{n+2}+2(512n^{16}+26752n^{15}+624800n^{14}+8677944n^{13}+80260596n^{12}+523718876n^{11}+2488583381n^{10}+8747566435n^9+22820793074n^8+43766004538n^7+60004107039n^6+55047935941n^5+27672902302n^4-778719870n^3-10812498240n^2-6360099840n-1300242000)a_{n+3}-3(n+4)(3n+8)(3n+10)(64n^{10}+1328n^9+11324n^8+52389n^7+143536n^6+233810n^5+204716n^4+48699n^3-68928n^2-61278n-15900)a_{n+4}=0$$


100
EURO
EYPΩ
EBPO
100