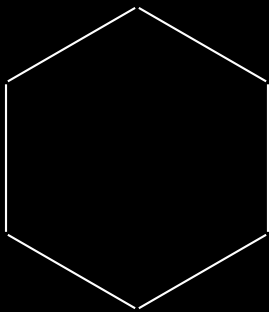
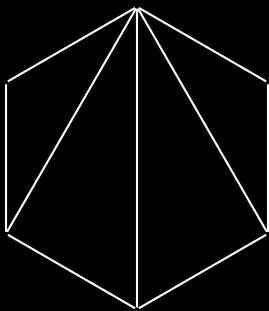


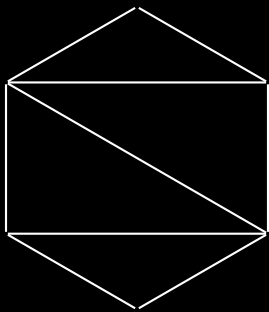
FLIP GRAPHS AND MATRIX MULTIPLICATION METHODS

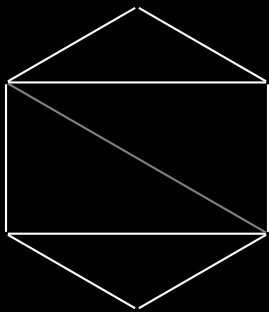


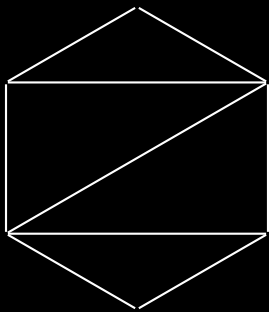
Manuel Kauers · Institute for Algebra · JKU

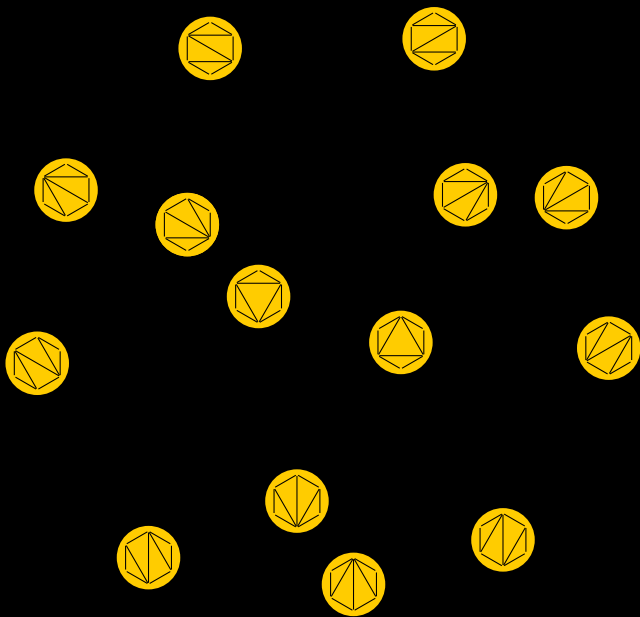


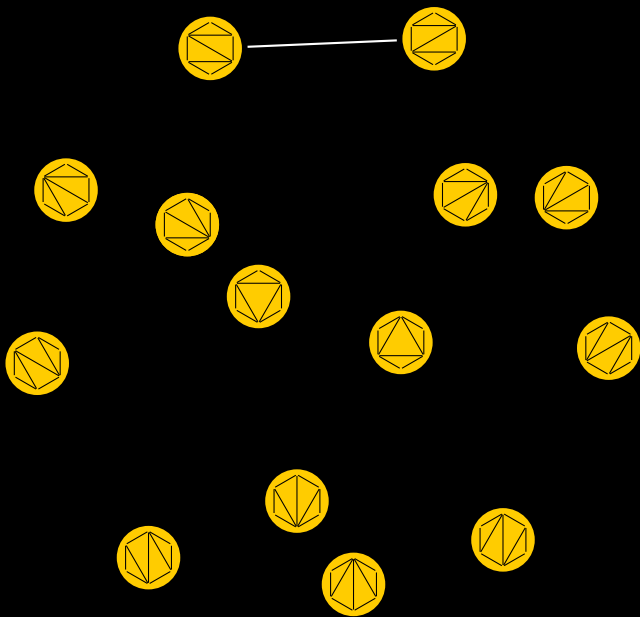


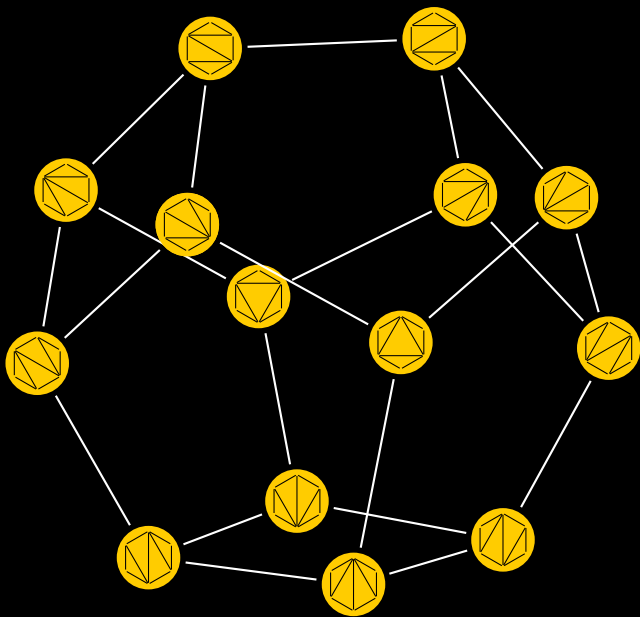












$$\begin{aligned}
& a_1 \otimes b_1 \otimes c_1 \\
+ & a_2 \otimes b_2 \otimes c_2 \\
+ & a_3 \otimes b_3 \otimes c_3 \\
+ & a_4 \otimes b_4 \otimes c_4 \\
+ & a_5 \otimes b_5 \otimes c_5
\end{aligned}$$

$$\begin{aligned}
& a_1 \otimes b_1 \otimes c_1 \\
+ & a_2 \otimes b_2 \otimes c_2 \\
+ & a_3 \otimes b_3 \otimes c_3 \\
+ & a_4 \otimes b_4 \otimes c_4 \\
+ & a_5 \otimes b_5 \otimes c_5
\end{aligned}$$

$$\begin{array}{ccc}
\mathfrak{M} & \mathfrak{M} & \mathfrak{M} \\
A & B & C
\end{array}$$

vector spaces

$$\begin{array}{r}
 a_1 \otimes b_1 \otimes c_1 \\
 + a_2 \otimes b_2 \otimes c_2 \\
 + a_3 \otimes b_3 \otimes c_3 \\
 + a_4 \otimes b_4 \otimes c_4 \\
 + a_5 \otimes b_5 \otimes c_5 \\
 \\
 \underbrace{\begin{array}{ccc} \mathfrak{M} & \mathfrak{M} & \mathfrak{M} \\ A & B & C \end{array}}_{\in A \otimes B \otimes C} \quad \text{vector spaces}
 \end{array}$$

$$\begin{array}{r}
 a_1 \otimes b_1 \otimes c_1 \\
 + a_2 \otimes b_2 \otimes c_2 \\
 + a_3 \otimes b_3 \otimes c_3 \\
 + a_4 \otimes b_4 \otimes c_4 \\
 + a_5 \otimes b_5 \otimes c_5
 \end{array}$$

a "row"



$$\underbrace{\begin{array}{ccc} \mathfrak{M} & \mathfrak{M} & \mathfrak{M} \\ A & B & C \end{array}}_{\in A \otimes B \otimes C}$$

vector spaces

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + \quad a \otimes b \otimes c'
 \end{array}$$

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + a \otimes b \otimes c'
 \end{array}
 =
 a \otimes b \otimes (c + c')$$

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + a \otimes b \otimes c'
 \end{array}
 =
 a \otimes b \otimes (c + c')$$

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + a \otimes b' \otimes c'
 \end{array}$$

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + a \otimes b \otimes c'
 \end{array}
 =
 a \otimes b \otimes (c + c')$$

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + a \otimes b' \otimes c'
 \end{array}
 =
 \begin{array}{c}
 a \otimes (b - b') \otimes c \\
 + a \otimes b' \otimes (c' + c)
 \end{array}$$

$$\begin{array}{c} a \otimes b \otimes c \\ + a \otimes b \otimes c' \end{array} = a \otimes b \otimes (c + c')$$

$$\begin{array}{c} a \otimes b \otimes c \\ + a \otimes b' \otimes c' \end{array} = \begin{array}{c} a \otimes b \otimes c \\ - a \otimes b' \otimes c \\ \uparrow \\ a \otimes (b - b') \otimes c \\ + a \otimes b' \otimes (c' + c) \\ \downarrow \\ a \otimes b' \otimes c' \\ + a \otimes b' \otimes c \end{array}$$

$$\begin{array}{c} a \otimes b \otimes c \\ + a \otimes b \otimes c' \end{array} = a \otimes b \otimes (c + c')$$

$$\begin{array}{c} a \otimes b \otimes c \\ + a \otimes b' \otimes c' \end{array} = \begin{array}{c} a \otimes b \otimes c \\ - a \otimes b' \otimes c \\ \uparrow \\ a \otimes (b - b') \otimes c \\ + a \otimes b' \otimes (c' + c) \\ \downarrow \\ a \otimes b' \otimes c' \\ + a \otimes b' \otimes c \end{array}$$

$$\begin{array}{c}
 a \otimes b \otimes c \\
 + a \otimes b \otimes c'
 \end{array}
 =
 a \otimes b \otimes (c + c')$$

Flip:

$$\begin{array}{c}
 \vdots \\
 + a \otimes b \otimes c \\
 + a \otimes b' \otimes c' \\
 \vdots
 \end{array}
 =
 \begin{array}{c}
 \vdots \\
 + a \otimes (b - b') \otimes c \\
 + a \otimes b' \otimes (c' + c) \\
 \vdots
 \end{array}$$

Reduction:

$$\begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{yellow}{c} \\ + \textcolor{green}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{red}{c}' \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes \textcolor{blue}{b} \otimes (\textcolor{yellow}{c} + \textcolor{red}{c}') \\ \vdots \end{array}$$

Flip:

$$\begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{yellow}{c} \\ + \textcolor{green}{a} \otimes \textcolor{red}{b}' \otimes \textcolor{yellow}{c}' \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes (\textcolor{blue}{b} - \textcolor{red}{b}') \otimes \textcolor{yellow}{c} \\ + \textcolor{green}{a} \otimes \textcolor{red}{b}' \otimes (\textcolor{yellow}{c}' + \textcolor{yellow}{c}) \\ \vdots \end{array}$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$a_0 \otimes (b_1 + b_0) \otimes c_1$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$a_0 \otimes (b_1 + b_0) \otimes c_1$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$a_0 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes b_1 \otimes (c_2 - c_1)$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$a_0 \otimes b_0 \otimes c_0$	$a_0 \otimes b_0 \otimes (c_0 - c_1)$
$a_0 \otimes b_1 \otimes c_1$	$a_0 \otimes (b_1 + b_0) \otimes c_1$
$a_1 \otimes b_0 \otimes c_1$	$a_1 \otimes (b_1 + b_0) \otimes c_1$
$a_1 \otimes b_1 \otimes c_2$	$a_1 \otimes b_1 \otimes (c_2 - c_1)$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$a_0 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes b_1 \otimes (c_2 - c_1)$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$(a_0 + a_1) \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes b_1 \otimes (c_2 - c_1)$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$a_0 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes b_1 \otimes (c_2 - c_1)$$

$$\begin{aligned} & a_0 \otimes b_0 \otimes (c_0 - c_1) \\ & (a_0 + a_1) \otimes (b_1 + b_0) \otimes c_1 \\ & a_1 \otimes b_1 \otimes (c_2 - c_1) \end{aligned}$$

Standard
algorithm

Karatsuba's
algorithm

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

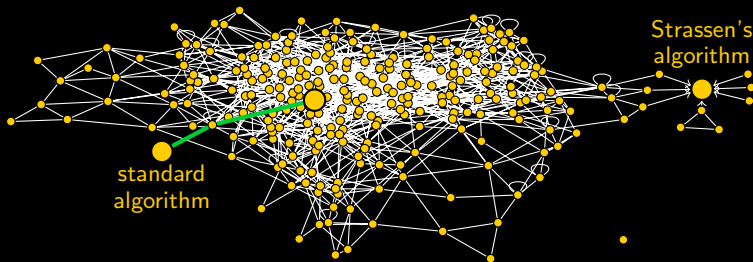
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



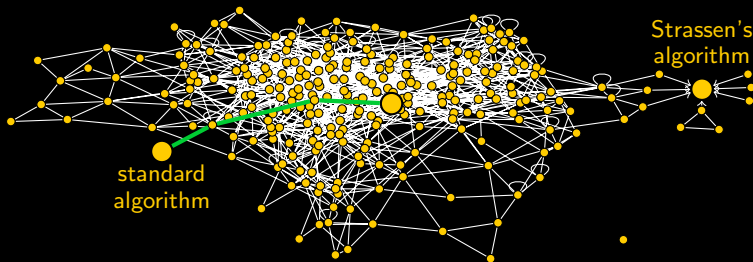
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



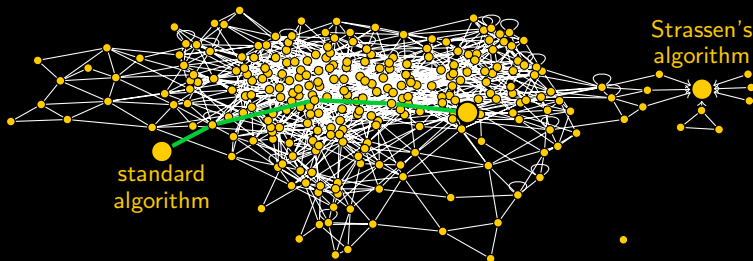
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



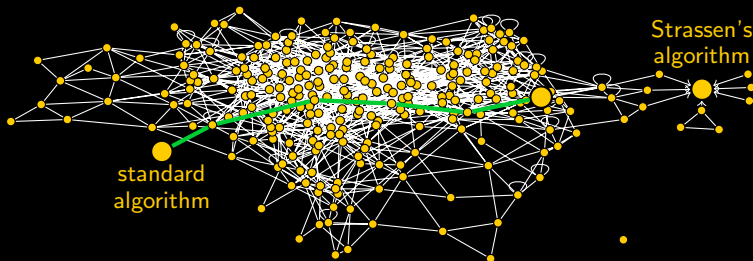
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



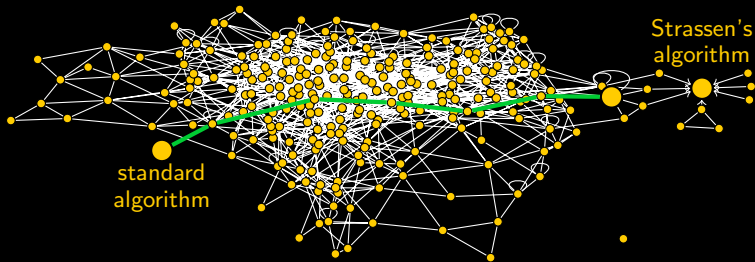
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



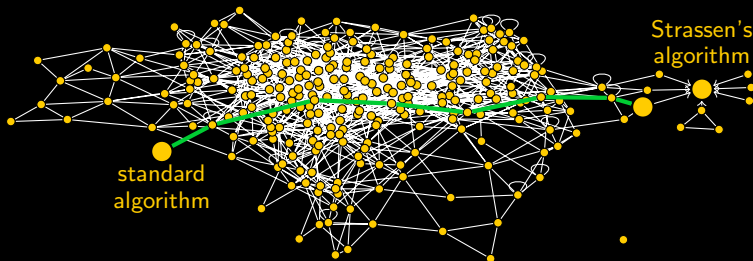
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



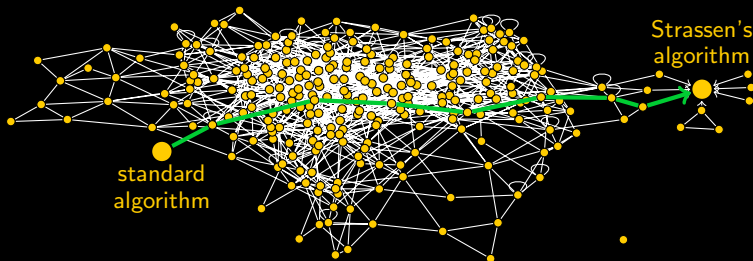
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

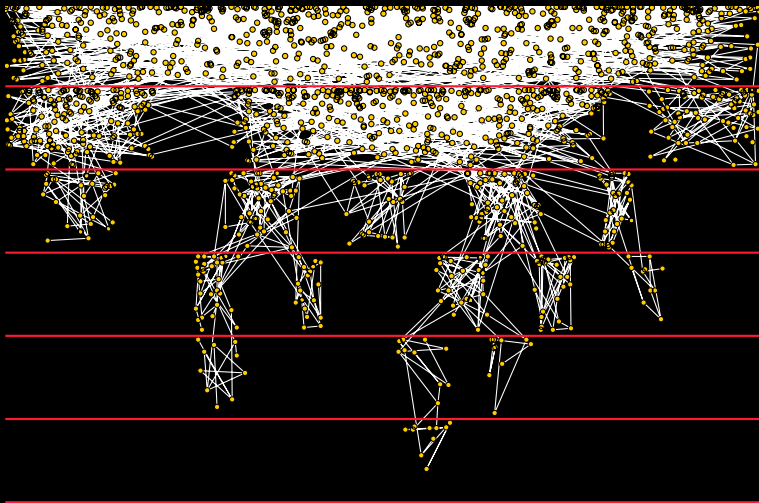


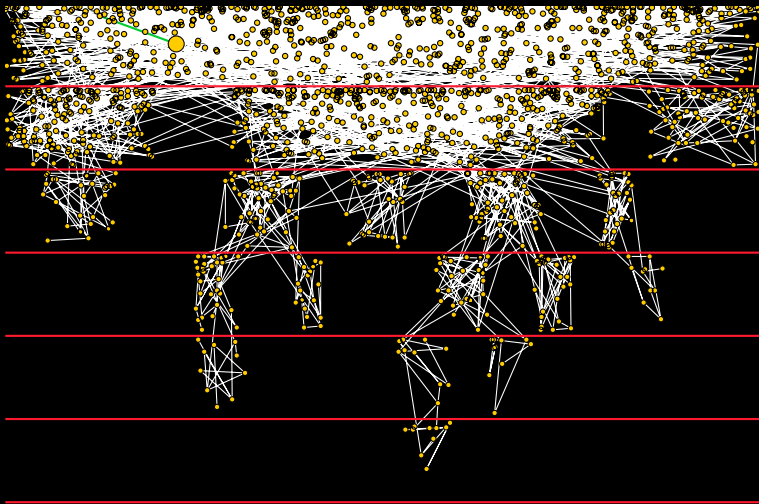
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$



$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$





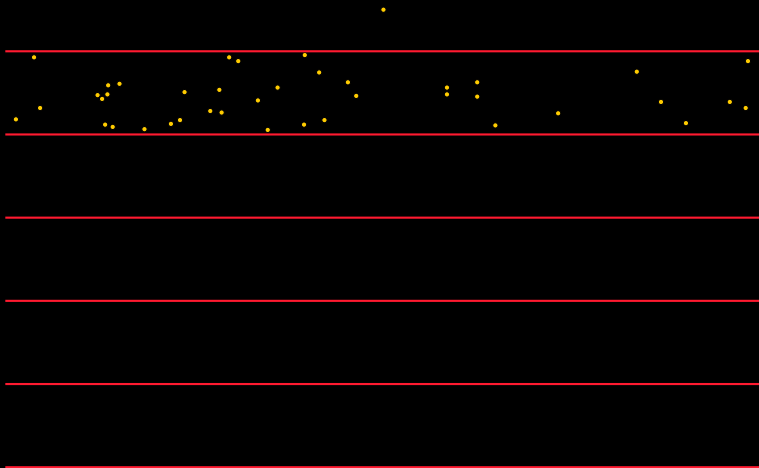


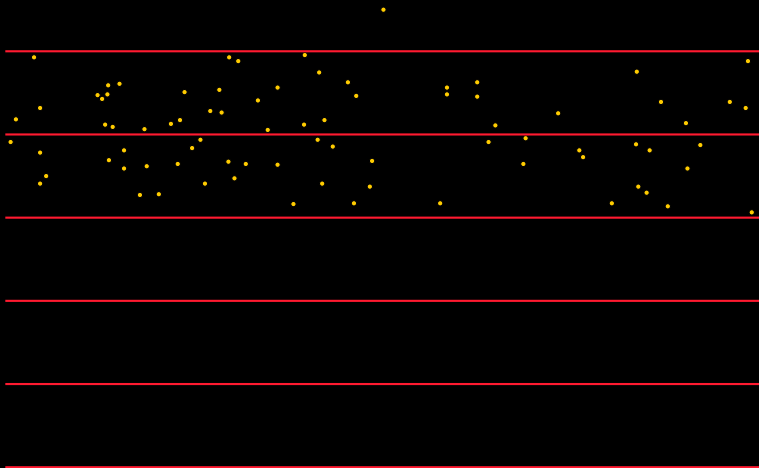
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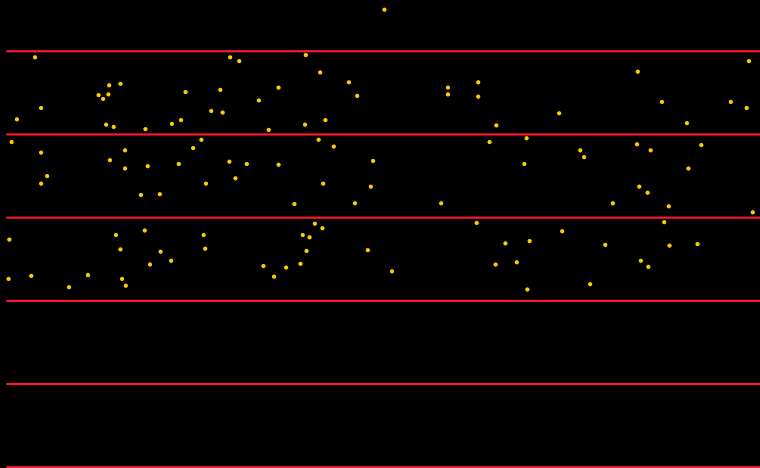
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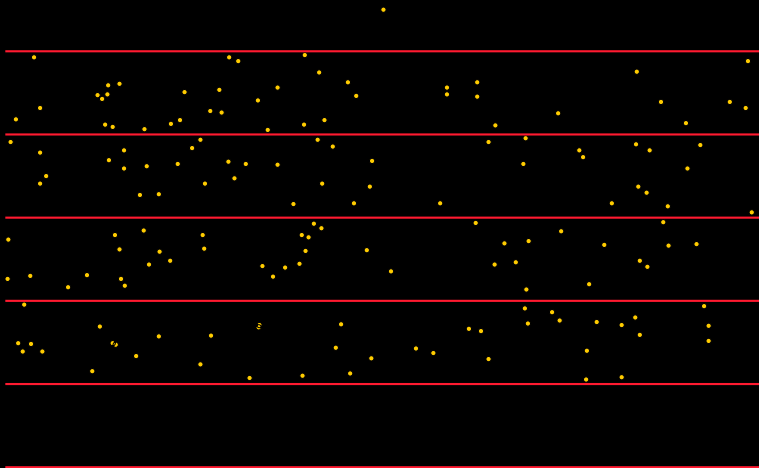
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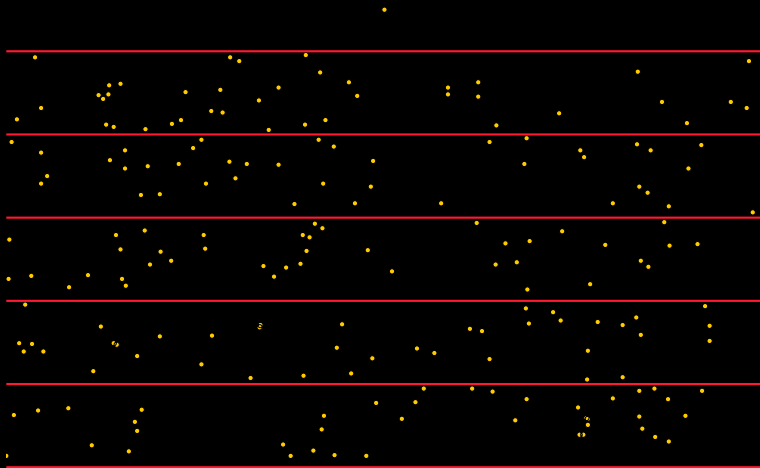
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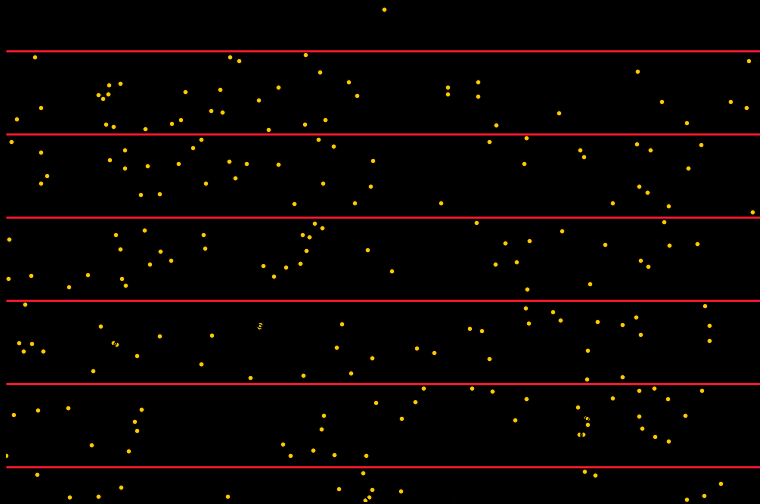












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$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

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$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$





$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

September
30
Friday



$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

October
1
Saturday



$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

October
1
Saturday



$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

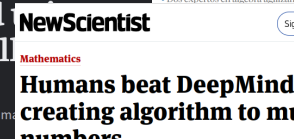
October
1
Saturday

Linzer Mathematiker übertrumpfen künstliche Intelligenz

Forscher der Johannes-Kepler-Universität fanden eine neue Methode, wie man besonders effizient 5×5 -Matrizen multipliziert. Sie stechen dabei Computermethoden aus

Reinhard Kleindl

19. Oktober 2022, 14:10

[135 Postings](#) [Später lesen](#)


来自奥地利林茨约翰·开普勒大学的研究人员 Manuel Kauers 和 Jakob Moosbauer 在其最新工作中表示，他们已经打破 AlphaTensor 的矩阵乘法记录。他们开发了一种以 95 步执行 5×5 矩阵乘法的方法，比 AlphaTensor 的 96 步记录少了一步，此前的记录为 98 步。论文预印版于 10 月 13 日发布在 arXiv 上。

Due ricercatori hanno battuto in matematica l'intelligenza artificiale di DeepMind

Risolto un problema matematico con un'operazione in meno rispetto all'intelligenza artificiale. Anche grazie a Google

di Luca Tremolada

17 ottobre 2022



ars TECHNICA

SECTIONS

FORUM



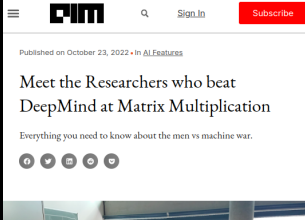
SIGN IN

IT'S GOING THE DISTANCE

DeepMind breaks 50-year math record with AI; new record falls a week later

AlphaTensor discovers better algorithms for matrix multiplication, inspiring another improvement from afar.

BENJ EDWARDS - OCT 13, 2022 1:46 PM 114



$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \end{pmatrix}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \end{pmatrix}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \end{pmatrix}$$







8



flip graph search



7



8



flip graph search

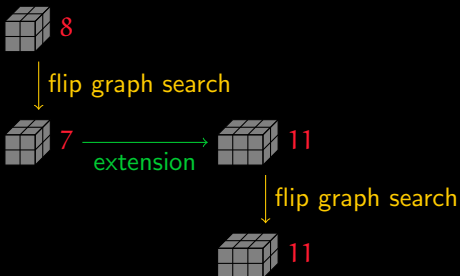


7

extension



11





8



flip graph search



7

extension



11



flip graph search

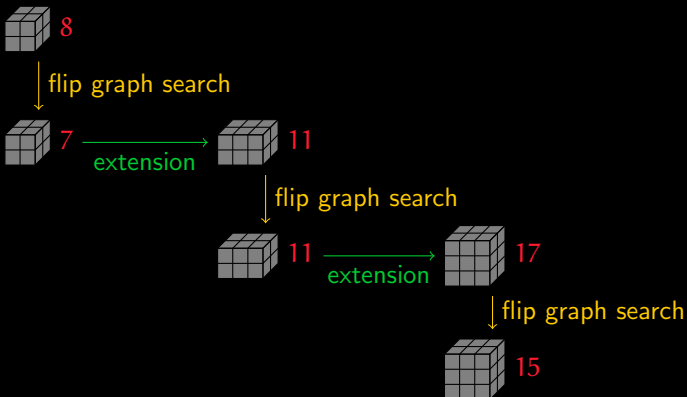


11

extension



17





8

↓ flip graph search



7

→ extension



11

↓ flip graph search



11

→ extension



17

↓ flip graph search

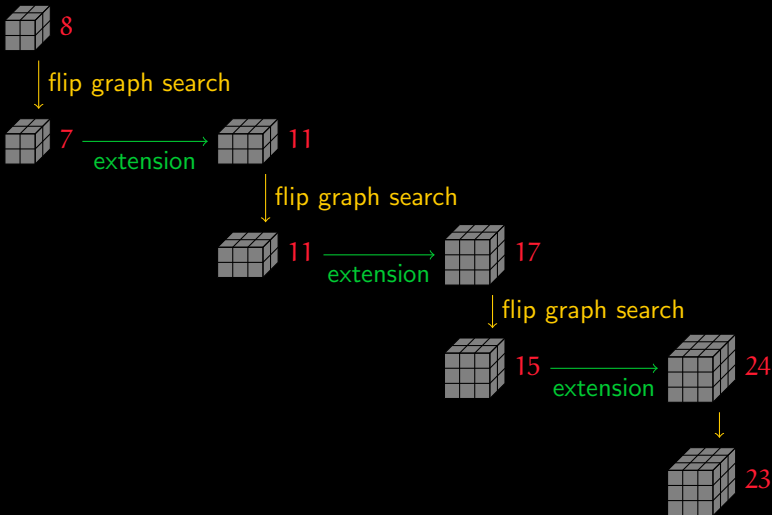


15

→ extension



24





flip graph search



extension



flip graph search



extension



flip graph search



extension



✓ much faster



8

↓ flip graph search



7

→ extension



11

↓ flip graph search



11

→ extension



17

↓ flip graph search



15

→ extension



24

↓



23

✓ much faster

✓ finds rank **94** for (5,5,5)



8

↓ flip graph search



7

→ extension



11

↓ flip graph search



11

→ extension



17

↓ flip graph search



15

→ extension



24

↓



23

✓ much faster

✓ finds rank **94** for (5,5,5)

Why stop here?

Collection of fast matrix multiplication algorithms - Chromium

Collection of fast matrix × +

← → ↻ 🔒 fmm.univ-lille.fr/index.html#view3x3x3

🌐 ↶ ☆ 🏠 👤 ⋮



Université
de Lille

ABOUT DATA BIBLIOGRAPHY CONTACT

Yet another catalogue of fast matrix multiplication algorithms.

Warning

Database under construction (including results of [Moosbauer and Poole 2025 \[arXiv\]](#) [Kauers and Wood 2025 \[arXiv\]](#),[\[arXiv\]](#) and [Novikov et al 2025 \(AlphaEvolve\) \[pdf\]](#)).

Front page is up to date but underlying data are not completely reconstructed.

As already done in:

- Robert L. Probert and Patrick Carl Fischer. *Decomposition techniques for matrix multiplication problems*. *Utilitas Mathematica*, 18:257–267, 1980.
- Warren D. Smith. *Fast matrix multiplication formulae*. Technical report, 2002.
- Marc Mezzarobba. *Autour de l'évaluation numérique des fonctions D-finies*. Master's thesis, École polytechnique, October 2011 [[.pdf](#)].
- Charles-Éric Drevet, Md. Nazrul Islam, and Éric Schost. *Optimization techniques for small matrix multiplication*. *Theoretical Computer Science*, 412(22):2219–2236, May 2011 [[DOI](#)].
- Alhussein Fawzi, Matej Balog, Aja Huang, Thomas Hubert, Bernardino Romera-Paredes, Mohammadamin Barekatian, Alexander Novikov, Francisco J.R. Ruiz, Julian Schrittwieser, Grzegorz Swirszcz, David Silver, Demis Hassabis, and Pushmeet Kohli. *Discovering faster matrix multiplication algorithms with reinforcement learning*. *Nature*, 610(7930):47–53, October 2022. [[DOI](#)]

Collection of fast matrix multiplication algorithms - Chromium

Collection of fast matrix multiplication algorithms - Chromium

fm.univ-lille.fr/index.html#view3x3x3

Version

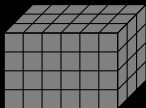
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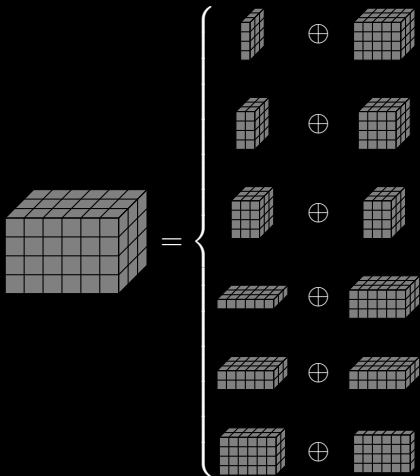
Upper bounds on non-commutative products in matrix multiplication

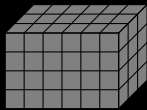
Algorithm	tensor rank	naive tensor rank	construction	ratio	complexity	stabilizer
(2×2×2:7)	7	8	Strassen 1969 [DOI]	.8750000000	2.807354922	$S_3 \times S_3$
(3×3×3:23)	23	27	Laderman 1976 [DOI]	.8518518518	2.854049830	$(C_2 \times C_2) \rtimes S_3$
(3×3×4:29)	29	36	Smirnov 2013 [DOI]	.8055555556	2.818985378	C_2
(3×3×5:36)	36	45	Smirnov 2013 [DOI]	.8000000000	2.824142367	C_1
(3×3×6:40)	40	54	Smirnov 2013 [DOI]	.7407407407	2.774299980	$((C_2 \times C_2) \rtimes S_4) \rtimes C_2$
(3×3×7:49)	49	63	(3×3×1:9) + (3×3×6:40)	.7777777778	2.818025883	C_1
(3×3×8:55)	55	72	(3×3×6:40) + (3×3×2:15)	.7638888889	2.811068066	C_1
(3×3×9:63)	63	81	(3×3×3:23) + (3×3×6:40)	.7777777778	2.828432812	C_1
(3×3×10:69)	69	90	(3×3×4:29) + (3×3×6:40)	.7666666667	2.822857064	C_1
(3×3×11:76)	76	99	(3×3×5:36) + (3×3×6:40)	.7676767677	2.827390894	C_1
(3×3×12:80)	80	108	(1×1×2:2) $\otimes$ (3×3×6:40)	.7407407407	2.807712827	$C_2 \times (C_2 \times C_2) \rtimes S_4$
(3×3×13:89)	89	117	(3×3×1:9) + (3×3×12:80)	.7606837607	2.827681075	C_1

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{1,4} \end{pmatrix}$$

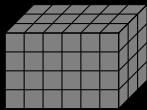




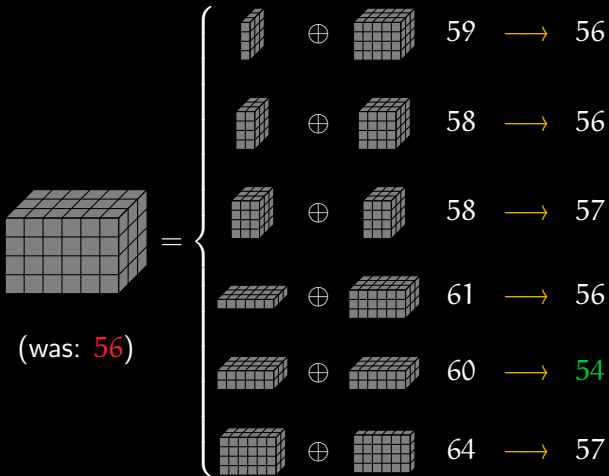




$$= \left\{ \begin{array}{lll} \begin{array}{c} \text{4x2x2 cube} \end{array} \oplus \begin{array}{c} \text{4x4x2 cube} \end{array} & 59 \\ \begin{array}{c} \text{2x2x4 cube} \end{array} \oplus \begin{array}{c} \text{4x4x2 cube} \end{array} & 58 \\ \begin{array}{c} \text{2x4x2 cube} \end{array} \oplus \begin{array}{c} \text{4x4x2 cube} \end{array} & 58 \\ \begin{array}{c} \text{4x4x1 slab} \end{array} \oplus \begin{array}{c} \text{4x4x3 cube} \end{array} & 61 \\ \begin{array}{c} \text{4x2x2 cube} \end{array} \oplus \begin{array}{c} \text{4x4x2 cube} \end{array} & 60 \\ \begin{array}{c} \text{4x4x2 cube} \end{array} \oplus \begin{array}{c} \text{4x4x2 cube} \end{array} & 64 \end{array} \right.$$



$$= \left\{ \begin{array}{llll} \begin{array}{c} \text{1x4x4 cube} \end{array} \oplus \begin{array}{c} \text{4x4x4 cube} \end{array} & 59 & \longrightarrow & 56 \\ \begin{array}{c} \text{2x2x4 cube} \end{array} \oplus \begin{array}{c} \text{4x4x4 cube} \end{array} & 58 & \longrightarrow & 56 \\ \begin{array}{c} \text{2x4x4 cube} \end{array} \oplus \begin{array}{c} \text{2x4x4 cube} \end{array} & 58 & \longrightarrow & 57 \\ \begin{array}{c} \text{4x2x4 cube} \end{array} \oplus \begin{array}{c} \text{4x4x4 cube} \end{array} & 61 & \longrightarrow & 56 \\ \begin{array}{c} \text{4x4x2 cube} \end{array} \oplus \begin{array}{c} \text{4x4x4 cube} \end{array} & 60 & \longrightarrow & 54 \\ \begin{array}{c} \text{4x4x4 cube} \end{array} \oplus \begin{array}{c} \text{4x4x4 cube} \end{array} & 64 & \longrightarrow & 57 \end{array} \right.$$



 : 56
↓
54

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55

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64

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73

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73

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79

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90

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104

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108

 : 115
↓
111

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129

 : 149
↓
145

Symmetries

$$\begin{aligned}
& a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
& + a_{2,2} \otimes b_{2,2} \otimes c_{2,2} \\
& + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
& + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
& + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
& + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
& + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{2,1} \otimes b_{1,2} \otimes c_{2,2}
\end{aligned}$$

$$\begin{array}{c}
 \left(\begin{array}{l}
 a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
 + a_{2,2} \otimes b_{2,2} \otimes c_{2,2} \\
 + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
 + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
 + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
 + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
 + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
 + a_{2,1} \otimes b_{1,2} \otimes c_{2,2}
 \end{array} \right)
 \end{array}$$

$$\begin{aligned}
& a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \quad \textcolor{blue}{P} \\
& + a_{2,2} \otimes b_{2,2} \otimes c_{2,2} \quad \textcolor{blue}{P} \\
& \left(\begin{array}{l} + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\ + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\ + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \end{array} \right) \textcolor{blue}{\left. \vphantom{\begin{array}{l} + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\ + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\ + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \end{array}} \right\}} \\
& \left(\begin{array}{l} + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\ + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\ + a_{2,1} \otimes b_{1,2} \otimes c_{2,2} \end{array} \right) \textcolor{blue}{\left. \vphantom{\begin{array}{l} + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\ + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\ + a_{2,1} \otimes b_{1,2} \otimes c_{2,2} \end{array}} \right\}}
\end{aligned}$$

$$\begin{aligned}
 & a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
 & + a_{2,2} \otimes b_{2,2} \otimes c_{2,2} \\
 & + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
 & + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
 & + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
 & + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
 & + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
 & + a_{2,1} \otimes b_{1,2} \otimes c_{2,2}
 \end{aligned}$$

$$\begin{aligned} & a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\ & + a_{2,2} \otimes b_{2,2} \otimes c_{2,2} \end{aligned}$$

$$\begin{aligned} & + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\ & + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\ & + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\ & + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\ & + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\ & + a_{2,1} \otimes b_{1,2} \otimes c_{2,2} \end{aligned}$$

$$+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$$

$$- (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$$

$$\begin{array}{l}
\boxed{
\begin{array}{l}
a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
+ a_{2,2} \otimes b_{2,2} \otimes c_{2,2}
\end{array}
} \\
\boxed{
\begin{array}{l}
+ a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
+ a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
+ a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
+ a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
+ a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
+ a_{2,1} \otimes b_{1,2} \otimes c_{2,2}
\end{array}
} \\
+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})
\end{array}$$

$$\begin{array}{l}
- a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
- a_{2,2} \otimes b_{2,2} \otimes c_{2,2} \\
- a_{1,1} \otimes b_{1,1} \otimes c_{2,2} \\
- a_{2,2} \otimes b_{1,1} \otimes c_{1,1} \\
- a_{1,1} \otimes b_{2,2} \otimes c_{1,1} \\
- a_{2,2} \otimes b_{2,2} \otimes c_{1,1} \\
- a_{1,1} \otimes b_{2,2} \otimes c_{2,2} \\
- a_{2,2} \otimes b_{1,1} \otimes c_{2,2}
\end{array}$$

$$\begin{aligned} & \mathbf{a}_{1,1} \otimes \mathbf{b}_{1,1} \otimes \mathbf{c}_{1,1} \\ + & \mathbf{a}_{2,2} \otimes \mathbf{b}_{2,2} \otimes \mathbf{c}_{2,2} \end{aligned}$$

$$\begin{aligned} + & \mathbf{a}_{1,1} \otimes \mathbf{b}_{1,2} \otimes \mathbf{c}_{2,1} \\ + & \mathbf{a}_{2,1} \otimes \mathbf{b}_{1,1} \otimes \mathbf{c}_{1,2} \\ + & \mathbf{a}_{1,2} \otimes \mathbf{b}_{2,1} \otimes \mathbf{c}_{1,1} \\ + & \mathbf{a}_{2,2} \otimes \mathbf{b}_{2,1} \otimes \mathbf{c}_{1,2} \\ + & \mathbf{a}_{1,2} \otimes \mathbf{b}_{2,2} \otimes \mathbf{c}_{2,1} \\ + & \mathbf{a}_{2,1} \otimes \mathbf{b}_{1,2} \otimes \mathbf{c}_{2,2} \end{aligned}$$

$$\begin{aligned} - & \mathbf{a}_{1,1} \otimes \mathbf{b}_{1,1} \otimes \mathbf{c}_{1,1} \\ - & \mathbf{a}_{2,2} \otimes \mathbf{b}_{2,2} \otimes \mathbf{c}_{2,2} \end{aligned}$$

$$\begin{aligned} - & \mathbf{a}_{1,1} \otimes \mathbf{b}_{1,1} \otimes \mathbf{c}_{2,2} \\ - & \mathbf{a}_{2,2} \otimes \mathbf{b}_{1,1} \otimes \mathbf{c}_{1,1} \\ - & \mathbf{a}_{1,1} \otimes \mathbf{b}_{2,2} \otimes \mathbf{c}_{1,1} \\ - & \mathbf{a}_{2,2} \otimes \mathbf{b}_{2,2} \otimes \mathbf{c}_{1,1} \\ - & \mathbf{a}_{1,1} \otimes \mathbf{b}_{2,2} \otimes \mathbf{c}_{2,2} \\ - & \mathbf{a}_{2,2} \otimes \mathbf{b}_{1,1} \otimes \mathbf{c}_{2,2} \end{aligned}$$

$$+ (\mathbf{a}_{1,1} + \mathbf{a}_{2,2}) \otimes (\mathbf{b}_{1,1} + \mathbf{b}_{2,2}) \otimes (\mathbf{c}_{1,1} + \mathbf{c}_{2,2})$$

$$\begin{aligned}
& a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
& + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
& + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
& + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
& + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{2,1} \otimes b_{1,2} \otimes c_{2,2}
\end{aligned}$$

$$\begin{aligned}
& - a_{1,1} \otimes b_{1,1} \otimes c_{2,2} \\
& - a_{2,2} \otimes b_{1,1} \otimes c_{1,1} \\
& - a_{1,1} \otimes b_{2,2} \otimes c_{1,1} \\
& - a_{2,2} \otimes b_{2,2} \otimes c_{1,1} \\
& - a_{1,1} \otimes b_{2,2} \otimes c_{2,2} \\
& - a_{2,2} \otimes b_{1,1} \otimes c_{2,2}
\end{aligned}$$

$$+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$$

	flip!	
$a_{1,1} \otimes b_{1,2} \otimes c_{2,1}$	—	$- a_{1,1} \otimes b_{1,1} \otimes c_{2,2}$
$+ a_{2,1} \otimes b_{1,1} \otimes c_{1,2}$	—	$- a_{2,2} \otimes b_{1,1} \otimes c_{1,1}$
$+ a_{1,2} \otimes b_{2,1} \otimes c_{1,1}$	—	$- a_{1,1} \otimes b_{2,2} \otimes c_{1,1}$
$+ a_{2,2} \otimes b_{2,1} \otimes c_{1,2}$	—	$- a_{2,2} \otimes b_{2,2} \otimes c_{1,1}$
$+ a_{1,2} \otimes b_{2,2} \otimes c_{2,1}$	—	$- a_{1,1} \otimes b_{2,2} \otimes c_{2,2}$
$+ a_{2,1} \otimes b_{1,2} \otimes c_{2,2}$	—	$- a_{2,2} \otimes b_{1,1} \otimes c_{2,2}$

$+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$

$$\begin{aligned}
& a_{1,1} \otimes b_{1,2} \otimes (c_{2,1} + c_{2,2}) \\
& + (a_{2,1} + a_{2,2}) \otimes b_{1,1} \otimes c_{1,2} \\
& + a_{1,2} \otimes (b_{2,1} + b_{2,2}) \otimes c_{1,1} \\
& + a_{2,2} \otimes b_{2,1} \otimes (c_{1,1} + c_{1,2}) \\
& + (a_{1,1} + a_{1,2}) \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{2,1} \otimes (b_{1,1} + b_{1,2}) \otimes c_{2,2}
\end{aligned}$$

$$\begin{aligned}
& - a_{1,1} \otimes (b_{1,1} + b_{1,2}) \otimes c_{2,2} \\
& - a_{2,2} \otimes b_{1,1} \otimes (c_{1,1} + c_{1,2}) \\
& - (a_{1,1} + a_{1,2}) \otimes b_{2,2} \otimes c_{1,1} \\
& - a_{2,2} \otimes (b_{2,1} + b_{2,2}) \otimes c_{1,1} \\
& - a_{1,1} \otimes b_{2,2} \otimes (c_{2,1} + c_{2,2}) \\
& - (a_{2,1} + a_{2,2}) \otimes b_{1,1} \otimes c_{2,2}
\end{aligned}$$

$$+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$$

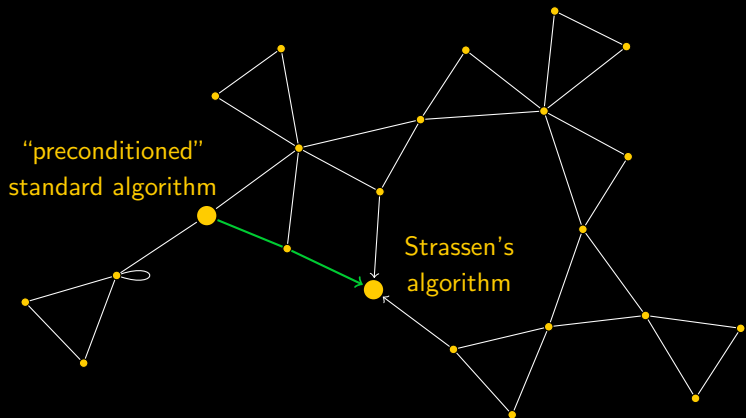
$$\begin{aligned}
& a_{1,1} \otimes b_{1,2} \otimes (c_{2,1} + c_{2,2}) \\
& + (a_{2,1} + a_{2,2}) \otimes b_{1,1} \otimes c_{1,2} \\
& + a_{1,2} \otimes (b_{2,1} + b_{2,2}) \otimes c_{1,1} \\
& + a_{2,2} \otimes b_{2,1} \otimes (c_{1,1} + c_{1,2}) \\
& + (a_{1,1} + a_{1,2}) \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{2,1} \otimes (b_{1,1} + b_{1,2}) \otimes c_{2,2}
\end{aligned}$$

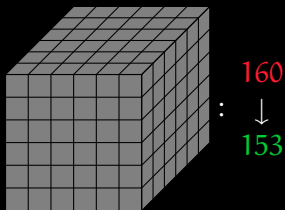
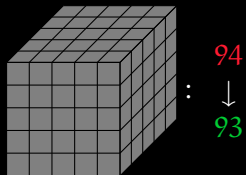
$$\begin{aligned}
& - a_{1,1} \otimes (b_{1,1} + b_{1,2}) \otimes c_{2,2} \\
& - a_{2,2} \otimes b_{1,1} \otimes (c_{1,1} + c_{1,2}) \\
& - (a_{1,1} + a_{1,2}) \otimes b_{2,2} \otimes c_{1,1} \\
& - a_{2,2} \otimes (b_{2,1} + b_{2,2}) \otimes c_{1,1} \\
& - a_{1,1} \otimes b_{2,2} \otimes (c_{2,1} + c_{2,2}) \\
& - (a_{2,1} + a_{2,2}) \otimes b_{1,1} \otimes c_{2,2}
\end{aligned}$$

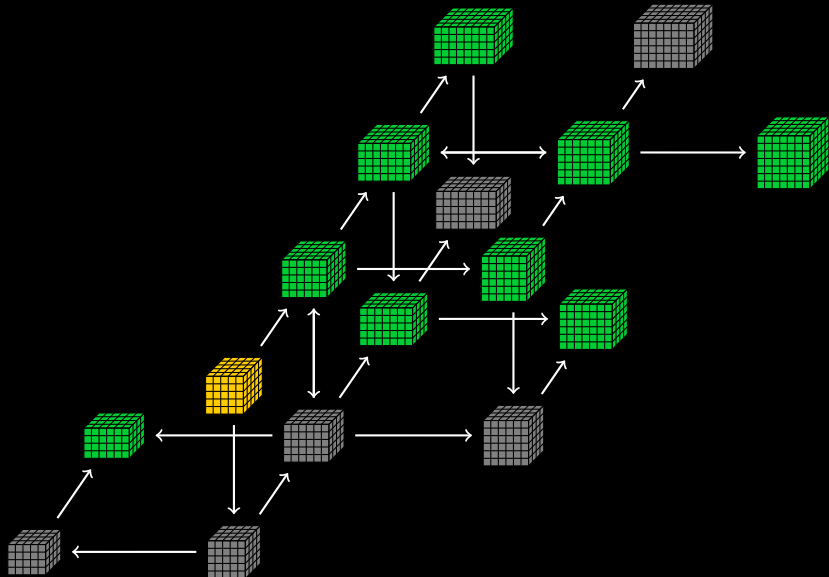
$$+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$$

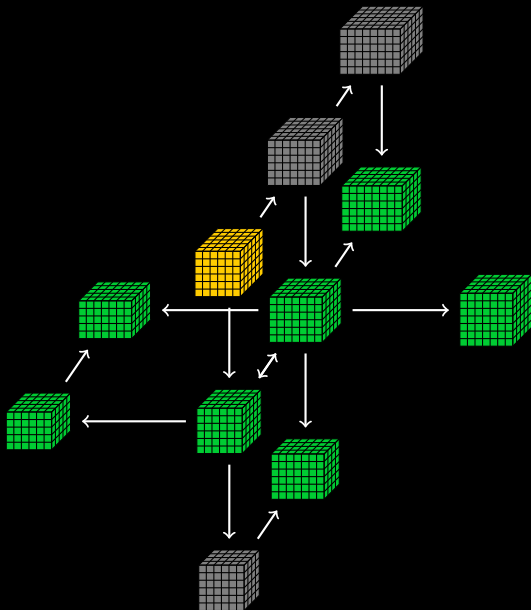
$$\begin{aligned}
& a_{1,1} \otimes (b_{1,2} - b_{2,2}) \otimes (c_{2,1} + c_{2,2}) \\
& + (a_{2,1} + a_{2,2}) \otimes b_{1,1} \otimes (c_{1,2} - c_{2,2}) \\
& + (a_{1,2} - a_{2,2}) \otimes (b_{2,1} + b_{2,2}) \otimes c_{1,1} \\
& + a_{2,2} \otimes (b_{2,1} - b_{1,1}) \otimes (c_{1,2} + c_{1,1}) \\
& + (a_{1,2} + a_{1,1}) \otimes b_{2,2} \otimes (c_{2,1} - c_{1,1}) \\
& + (a_{2,1} - a_{1,1}) \otimes (b_{1,2} + b_{1,1}) \otimes c_{2,2}
\end{aligned}$$

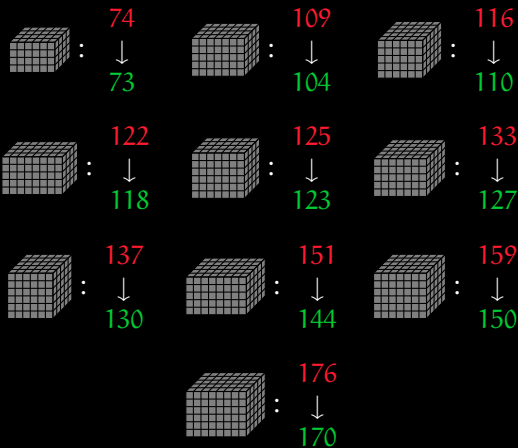
$$+ (a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})$$











The Ground Ring

Flip:

$$\begin{array}{c} \vdots \\ + \textcolor{teal}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{teal}{c} \\ + \textcolor{teal}{a} \otimes \textcolor{red}{b}' \otimes \textcolor{teal}{c}' \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + \textcolor{teal}{a} \otimes (\textcolor{blue}{b} - \textcolor{red}{b}') \otimes \textcolor{teal}{c} \\ + \textcolor{teal}{a} \otimes \textcolor{red}{b}' \otimes (\textcolor{teal}{c}' + \textcolor{teal}{c}) \\ \vdots \end{array}$$

Flip:

$$\begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{blue}{c} \\ + \textcolor{green}{a} \otimes \textcolor{red}{b}' \otimes \textcolor{red}{c}' \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes (\textcolor{blue}{b} - \boxed{\lambda} \textcolor{red}{b}') \otimes \textcolor{blue}{c} \\ + \textcolor{green}{a} \otimes \textcolor{red}{b}' \otimes (\textcolor{red}{c}' + \boxed{\lambda} \textcolor{blue}{c}) \\ \vdots \end{array}$$

Flip:

$$\begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{yellow}{c} \\ + \textcolor{green}{a} \otimes \textcolor{red}{b}' \otimes \textcolor{black}{c}' \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + \textcolor{green}{a} \otimes (\textcolor{blue}{b} - \boxed{\lambda} \textcolor{red}{b}') \otimes \textcolor{yellow}{c} \\ + \textcolor{green}{a} \otimes \textcolor{red}{b}' \otimes (\textcolor{black}{c}' + \boxed{\lambda} \textcolor{yellow}{c}) \\ \vdots \end{array}$$

How to choose $\boxed{\lambda}$?

😊 There is just one choice for $K = \mathbb{Z}_2$.

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- 😞 But schemes working for $K = \mathbb{Q}$ would be more interesting.

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- ☹ Sometimes.

$$\begin{aligned}
& \sum_{\ell=1}^{23} \sum_{i_1, i_2, j_1, j_2, k_1, k_2=1}^3 \alpha_{i_1, i_2}^\ell \beta_{j_1, j_2}^\ell \gamma_{k_1, k_2}^\ell (a_{i_1, i_2} \otimes b_{j_1, j_2} \otimes c_{k_1, k_2}) \\
& \stackrel{!}{=} \sum_{i, j, k=1}^3 a_{i, j} \otimes b_{j, k} \otimes c_{k, i}
\end{aligned}$$

$\forall i_1, i_2, j_1, j_2, k_1, k_2 \in \{1, 2, 3\} :$

$$\sum_{\ell=1}^{23} \alpha_{i_1, i_2}^{\ell} \beta_{j_1, j_2}^{\ell} \gamma_{k_1, k_2}^{\ell} = \delta_{i_1, k_2} \delta_{j_1, i_2} \delta_{k_1, j_2}$$

$\forall i_1, i_2, j_1, j_2, k_1, k_2 \in \{1, 2, 3\} :$

$$\sum_{\ell=1}^{23} \alpha_{i_1, i_2}^{\ell} \beta_{j_1, j_2}^{\ell} \gamma_{k_1, k_2}^{\ell} = \delta_{i_1, k_2} \delta_{j_1, i_2} \delta_{k_1, j_2}$$

$23 \cdot 3 \cdot 3 \cdot 3$ variables and 3^6 equations

$$5\alpha^2 - 23\alpha + 12 = 0: \text{ one variable, one equation}$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1\alpha_0 + 2\alpha_1 + 4\alpha_2 + 8\alpha_3 + 16\alpha_4 + 32\alpha_5 + \cdots$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1\alpha_0 + 2\alpha_1 + 4\alpha_2 + 8\alpha_3 + 16\alpha_4 + 32\alpha_5 + \cdots$$

$$\underbrace{\hspace{1.5cm}}_{=(\bmod 2)}$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1\alpha_0 + 2\alpha_1 + 4\alpha_2 + 8\alpha_3 + 16\alpha_4 + 32\alpha_5 + \cdots$$

$$\underbrace{\hspace{10em}}_{=(\bmod 2^2)}$$

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$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1\alpha_0 + 2\alpha_1 + 4\alpha_2 + 8\alpha_3 + 16\alpha_4 + 32\alpha_5 + \dots$$

$$\underbrace{\hspace{10em}}_{=(\text{mod } 2^4)}$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1\alpha_0 + 2\alpha_1 + 4\alpha_2 + 8\alpha_3 + 16\alpha_4 + 32\alpha_5 + \cdots$$

$$\underbrace{\hspace{10em}}_{= (\text{mod } 2^5)}$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1\alpha_0 + 2\alpha_1 + 4\alpha_2 + 8\alpha_3 + 16\alpha_4 + 32\alpha_5 + \dots$$

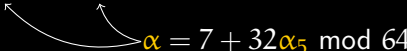
$$\underbrace{\hspace{15em}}_{= (\text{mod } 2^6)}$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 + 8 \cdot 0 + 16 \cdot 0 + 32 \cdot \alpha_5 + \dots$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 7 + 32\alpha_5 \bmod 64$$

$$5\alpha^2 - 23\alpha + 12 = 0: \text{ one variable, one equation}$$


The diagram illustrates the substitution of a congruence relation into a quadratic equation. Two curved arrows originate from the expression $\alpha = 7 + 32\alpha_5 \pmod{64}$. One arrow points to the α^2 term in the equation $5\alpha^2 - 23\alpha + 12 = 0$, and the other points to the α term in the same equation.

$$\alpha = 7 + 32\alpha_5 \pmod{64}$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation



$\alpha = 7 + 32\alpha_5 \bmod 64$

$$5(7 + 32\alpha_5)^2 - 23(7 + 32\alpha_5) + 12 = 0 \bmod 64$$

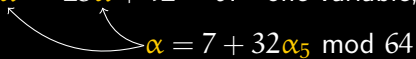
$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation



$\alpha = 7 + 32\alpha_5 \bmod 64$

$$32 + 32\alpha_5 = 0 \bmod 64$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation



$\alpha = 7 + 32\alpha_5 \bmod 64$

The diagram shows two curved arrows originating from the expression $\alpha = 7 + 32\alpha_5 \bmod 64$. One arrow points to the α^2 term in the equation above, and the other points to the α term in the same equation.

$$1 + \alpha_5 = 0 \bmod 2$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation



$\alpha = 7 + 32\alpha_5 \bmod 64$

$$\alpha_5 = 1 \bmod 2$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 39 \bmod 64$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 103 \bmod 128$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 103 \bmod 256$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 103 \bmod 512$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 615 \bmod 1024$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1639 \bmod 2048$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1639 \bmod 2048$$

$$q\alpha = 1639q \bmod 2048$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

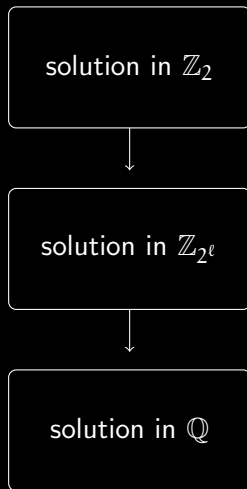
$$\alpha = 1639 \bmod 2048$$

$$5\alpha = 3 \bmod 2048$$

$5\alpha^2 - 23\alpha + 12 = 0$: one variable, one equation

$$\alpha = 1639 \bmod 2048$$

$$5\alpha = 3$$



Flip graphs

solution in $\mathbb{Z}_2$



Hensel
lifting

solution in $\mathbb{Z}_{2^l}$



Rational
reconstruction

solution in $\mathbb{Q}$



Flip graphs

solution in $\mathbb{Z}_2$



Hensel
lifting

solution in $\mathbb{Z}_{2^\ell}$

might
not
exist



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Flip graphs

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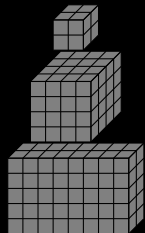


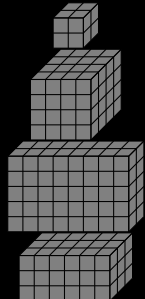
Rational
reconstruction

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exist



format	best rank	best ground domain	source
	7	$\mathbb{Z}$	Strassen
	47	$\mathbb{Z}_2$	AlphaTensor
	63	$\mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \frac{1}{7}]$	Flip graphs

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	40	$\mathbb{Z}[\frac{1}{2}]$	Smirnov

Other Tensors

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix}$$

$$\begin{aligned}
& a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
& + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
& + a_{1,3} \otimes b_{3,1} \otimes c_{1,1} \\
& + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
& + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
& + a_{2,3} \otimes b_{3,1} \otimes c_{1,2} \\
& + a_{3,1} \otimes b_{1,1} \otimes c_{1,3} \\
& + a_{3,2} \otimes b_{2,1} \otimes c_{1,3} \\
& + a_{3,3} \otimes b_{3,1} \otimes c_{1,3} \\
& + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
& + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{1,3} \otimes b_{3,2} \otimes c_{2,1} \\
& + \dots
\end{aligned}$$

$$\begin{aligned}
& a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
& + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
& + a_{1,3} \otimes b_{3,1} \otimes c_{1,1} \\
& + a_{2,1} \otimes b_{1,1} \otimes c_{1,2} \\
& + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
& + a_{2,3} \otimes b_{3,1} \otimes c_{1,2} \\
& + a_{3,1} \otimes b_{1,1} \otimes c_{1,3} \\
& + a_{3,2} \otimes b_{2,1} \otimes c_{1,3} \\
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& + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
& + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{1,3} \otimes b_{3,2} \otimes c_{2,1} \\
& + \dots
\end{aligned}$$

$$\begin{aligned}
& a_{1,1} \otimes b_{1,1} \otimes c_{1,1} \\
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& + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
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& + a_{3,1} \otimes b_{1,1} \otimes c_{1,3} \\
& + a_{3,2} \otimes b_{2,1} \otimes c_{1,3} \\
& + a_{3,3} \otimes b_{3,1} \otimes c_{1,3} \\
& + a_{1,1} \otimes b_{1,2} \otimes c_{2,1} \\
& + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{1,3} \otimes b_{3,2} \otimes c_{2,1} \\
& + \dots
\end{aligned}$$

$$\begin{aligned}
& b_{1,1} \otimes a_{1,1} \otimes c_{1,1} \\
& + a_{1,2} \otimes b_{2,1} \otimes c_{1,1} \\
& + a_{1,3} \otimes b_{3,1} \otimes c_{1,1} \\
& + b_{1,1} \otimes a_{2,1} \otimes c_{1,2} \\
& + a_{2,2} \otimes b_{2,1} \otimes c_{1,2} \\
& + a_{2,3} \otimes b_{3,1} \otimes c_{1,2} \\
& + b_{1,1} \otimes a_{3,1} \otimes c_{1,3} \\
& + a_{3,2} \otimes b_{2,1} \otimes c_{1,3} \\
& + a_{3,3} \otimes b_{3,1} \otimes c_{1,3} \\
& + b_{1,2} \otimes a_{1,1} \otimes c_{2,1} \\
& + a_{1,2} \otimes b_{2,2} \otimes c_{2,1} \\
& + a_{1,3} \otimes b_{3,2} \otimes c_{2,1} \\
& + \dots
\end{aligned}$$

$$\begin{array}{c}
\vdots \\
+ \textcolor{blue}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{blue}{c} \\
+ \textcolor{blue}{a}' \otimes \textcolor{blue}{b}' \otimes \textcolor{blue}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ (\textcolor{blue}{a} - \textcolor{blue}{a}') \otimes \textcolor{blue}{b} \otimes \textcolor{blue}{c} \\
+ \textcolor{blue}{a}' \otimes (\textcolor{blue}{b} + \textcolor{blue}{b}') \otimes \textcolor{blue}{c} \\
\vdots
\end{array}$$

$$\begin{array}{c} \vdots \\ + \textcolor{blue}{a} \otimes (\textcolor{red}{b} - \textcolor{red}{b}') \otimes \textcolor{green}{c} \\ + (\textcolor{blue}{a} + \textcolor{blue}{a}') \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + \textcolor{blue}{a} \otimes \textcolor{red}{b} \otimes \textcolor{green}{c} \\ + \textcolor{blue}{a}' \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\ \vdots \end{array} = \begin{array}{c} \vdots \\ + (\textcolor{blue}{a} - \textcolor{blue}{a}') \otimes \textcolor{red}{b} \otimes \textcolor{green}{c} \\ + \textcolor{blue}{a}' \otimes (\textcolor{red}{b} + \textcolor{red}{b}') \otimes \textcolor{green}{c} \\ \vdots \end{array}$$

$$\begin{array}{c}
\vdots \\
+ \textcolor{blue}{a} \otimes (\textcolor{blue}{b} - \textcolor{red}{b}') \otimes \textcolor{green}{c} \\
+ (\textcolor{blue}{a} + \textcolor{red}{a}') \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ \textcolor{blue}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ (\textcolor{blue}{a} - \textcolor{red}{a}') \otimes \textcolor{blue}{b} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes (\textcolor{blue}{b} + \textcolor{red}{b}') \otimes \textcolor{green}{c} \\
\vdots
\end{array}$$

$$=
\begin{array}{c}
\vdots \\
+ \textcolor{blue}{b} \otimes \textcolor{blue}{a} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}$$

$$\begin{array}{c}
\vdots \\
+ \textcolor{blue}{a} \otimes (\textcolor{blue}{b} - \textcolor{red}{b}') \otimes \textcolor{green}{c} \\
+ (\textcolor{blue}{a} + \textcolor{red}{a}') \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ \textcolor{blue}{a} \otimes \textcolor{blue}{b} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ (\textcolor{blue}{a} - \textcolor{red}{a}') \otimes \textcolor{blue}{b} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes (\textcolor{blue}{b} + \textcolor{red}{b}') \otimes \textcolor{green}{c} \\
\vdots
\end{array}$$

$$=$$

$$\begin{array}{c}
\vdots \\
+ \textcolor{blue}{b} \otimes (\textcolor{blue}{a} - \textcolor{red}{b}') \otimes \textcolor{green}{c} \\
+ (\textcolor{blue}{b} + \textcolor{red}{a}') \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ \textcolor{blue}{b} \otimes \textcolor{blue}{a} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes \textcolor{red}{b}' \otimes \textcolor{green}{c} \\
\vdots
\end{array}
=
\begin{array}{c}
\vdots \\
+ (\textcolor{blue}{b} - \textcolor{red}{a}') \otimes \textcolor{blue}{a} \otimes \textcolor{green}{c} \\
+ \textcolor{red}{a}' \otimes (\textcolor{blue}{a} + \textcolor{red}{b}') \otimes \textcolor{green}{c} \\
\vdots
\end{array}$$

Theorem (Rosowski, JSC'23). Multiplying an $n \times m$ matrix with an $m \times p$ matrix over a **commutative** ground ring costs no more than

$$\begin{cases} (m(np + n + p - 1) + \min(n, p) - 1)/2 & \text{if } m \text{ is odd and } n, p \text{ are even} \\ m(np + n + p - 1)/2 & \text{otherwise} \end{cases}$$

multiplications in that ring.

Theorem (Rosowski, JSC'23). Multiplying an $n \times m$ matrix with an $m \times p$ matrix over a **commutative** ground ring costs no more than

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Flip graph searches match this bound for all $n, m, p \leq 5$.

No better multiplication scheme was found in this range.

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

$$a_0 \otimes b_0 \otimes c_0$$

$$a_0 \otimes b_1 \otimes c_1$$

$$a_1 \otimes b_0 \otimes c_1$$

$$a_1 \otimes b_1 \otimes c_2$$

$$a_0 \otimes b_0 \otimes (c_0 - c_1)$$

$$a_0 \otimes (b_1 + b_0) \otimes c_1$$

$$a_1 \otimes (b_1 + b_0) \otimes c_1$$

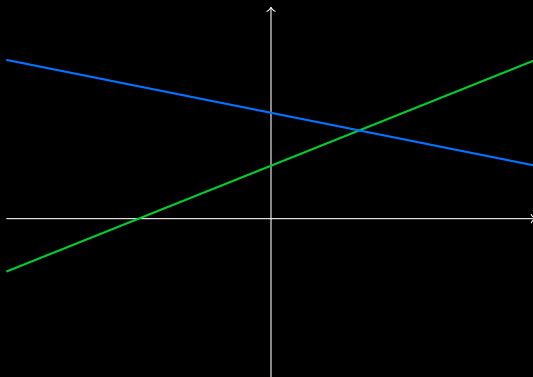
$$a_1 \otimes b_1 \otimes (c_2 - c_1)$$

$$\begin{aligned} & a_0 \otimes b_0 \otimes (c_0 - c_1) \\ & (a_0 + a_1) \otimes (b_1 + b_0) \otimes c_1 \\ & a_1 \otimes b_1 \otimes (c_2 - c_1) \end{aligned}$$

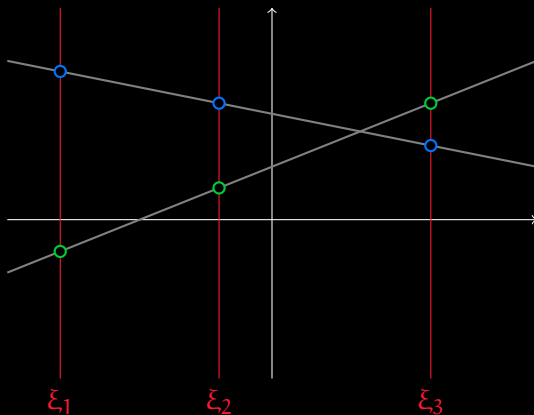
Standard
algorithm

Karatsuba's
algorithm

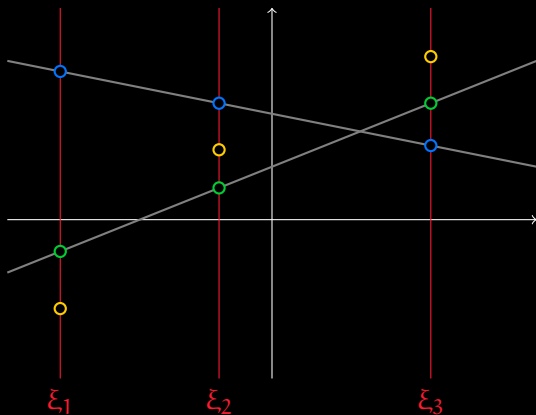
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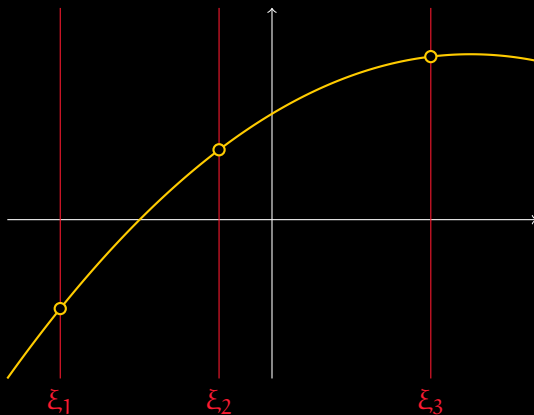
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$$\begin{aligned} & (a_0 + \xi_1 a_1 + \xi_1^2 a_2) \otimes (b_0 + \xi_1 b_1 + \xi_1^2 b_2) \otimes \frac{\xi_2 \xi_3 c_0 + (\xi_2 + \xi_3) c_1 + c_2}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} \\ & + (a_0 + \xi_2 a_1 + \xi_2^2 a_2) \otimes (b_0 + \xi_2 b_1 + \xi_2^2 b_2) \otimes \frac{\xi_1 \xi_3 c_0 + (\xi_1 + \xi_3) c_1 + c_2}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} \\ & + (a_0 + \xi_3 a_1 + \xi_3^2 a_2) \otimes (b_0 + \xi_3 b_1 + \xi_3^2 b_2) \otimes \frac{\xi_1 \xi_2 c_0 + (\xi_1 + \xi_2) c_1 + c_2}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} \end{aligned}$$

$$(a_0 + a_1x)(b_0 + b_1x) = c_0 + c_1x + c_2x^2$$

Fact: If the ground field is not too small, multiplication of a degree n polynomial with a degree m polynomial has tensor rank $n+m+1$.

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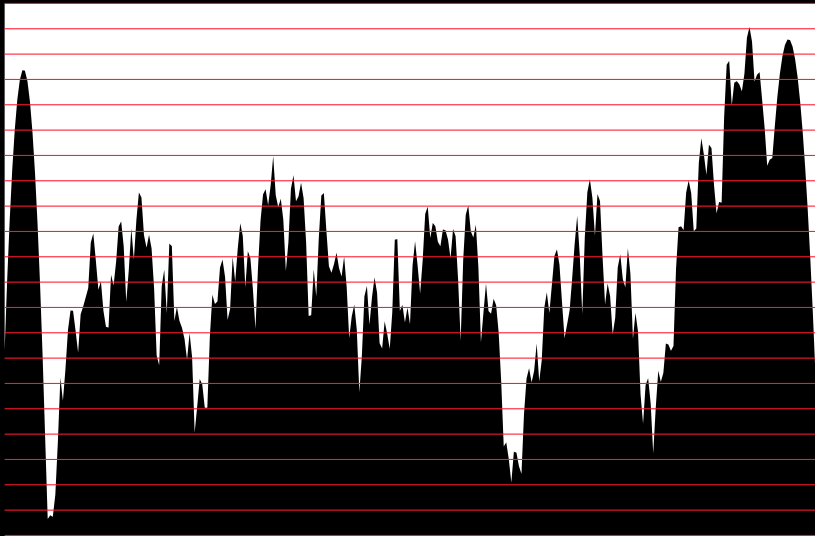
Question: Is this scheme reachable from the standard algorithm in the flip graph (without “going up”)?

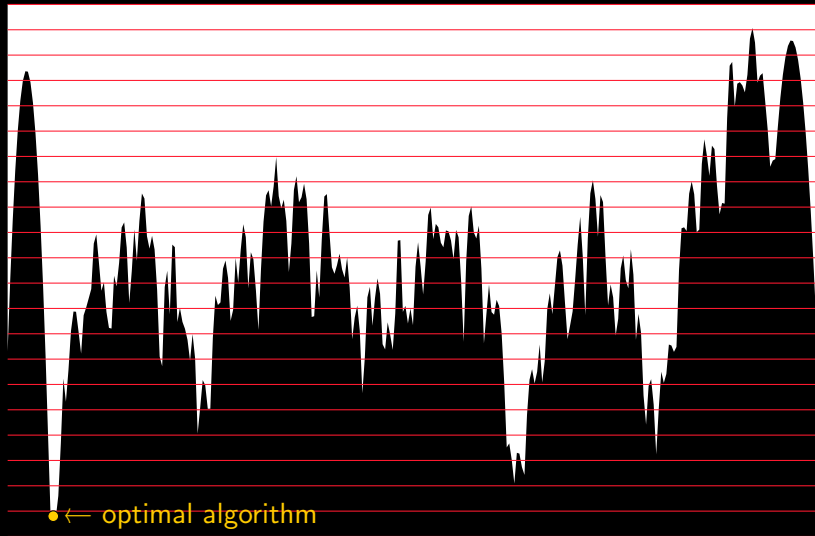
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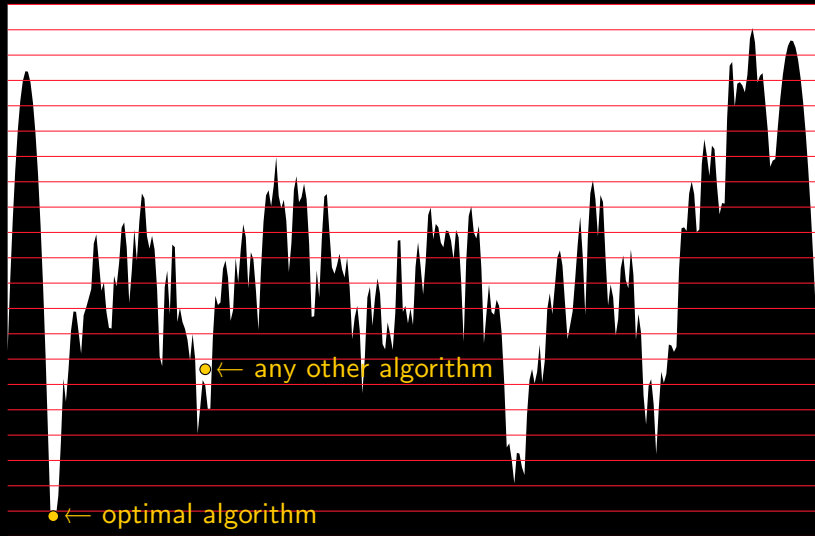
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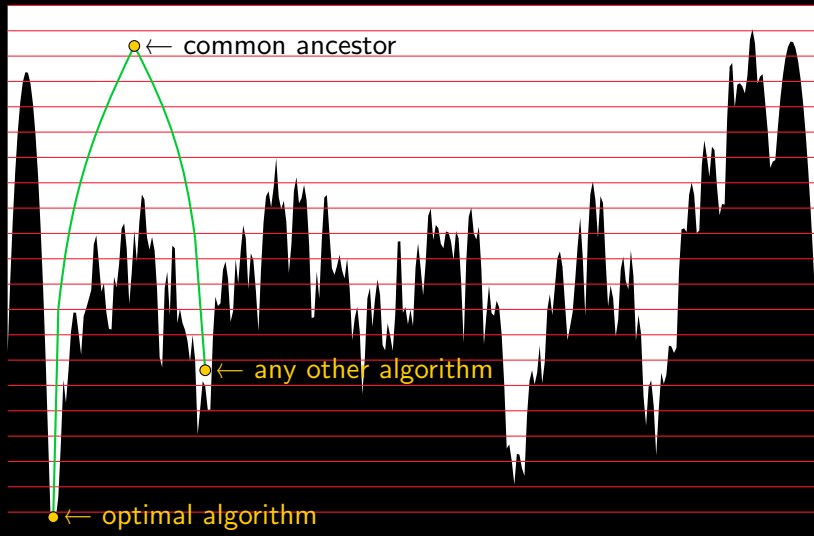
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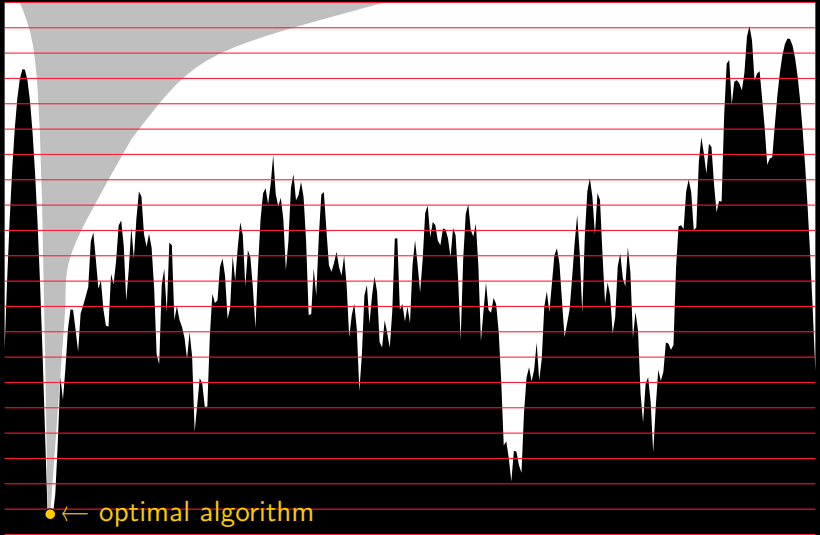
Answer: Yes.

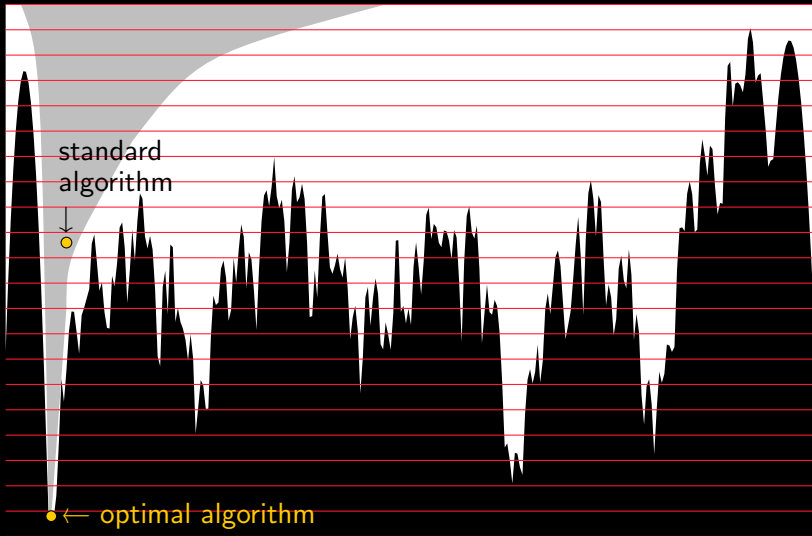












Is this also true for matrices?

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Can we traverse the flip graph more systematically?

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Which other tensors should we consider?