D-Finiteness: A Success Story

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ABSTRACT

A considerable portion of the work on special functions in computer algebra during the past decades was focused on D-finite functions. This focus was chosen for good reasons, as the concept of D-finiteness has proven to provide a fairly good compromise between, on the one hand, covering as many functions as possible, and on the other hand, keeping the class of functions restricted enough that computations stay reasonably efficient. In the talk, we will illustrate how questions about D-finite functions naturally arise in applications and how computer algebra is nowadays routinely used to answer such questions.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Differential equations, linear operators, symbolic summation and integration, special functions, binomial identities, computer algebra

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1 INTRODUCTION

It has taken a long time until it was realized that the roots of a polynomial cannot always be expressed in terms of radicals. Nowadays, we can construct for a given polynomial such representations of the roots whenever they exist, but even when this is the case, it is often more convenient to express the roots implicitly through the polynomial equations they satisfy. The situation is quite similar with ordinary linear differential equations with polynomial coefficients: their solutions need not admit any "closed form" expressions, and although have algorithms that can find such expressions whenever they exist (or prove that they don't exist), it is usually more convenient to use the differential equation itself for representing its solutions.

A (univariate) function is called *D-finite* if there is an ordinary linear differential equation with polynomial coefficients which has this functions among its solutions. The concept of D-finite functions is therefore conceptually similar to the concept of algebraic numbers. Many functions arising in all kinds of different contexts are D-finite. As already pointed out by Salvy in his invited talk at ISSAC'05 [77], "approximately 60% of the functions described in Abramowitz & Stegun's handbook [4] fall into this category". Moreover, many applications lead to D-finite functions that have not acquired a particular name.

There is also a discrete version of the concept: a (univariate) infinite sequence is called *P-recursive* (or *P-finite* or simply also *D-finite*) if it satisfies a linear recurrence equation with polynomial coefficients. Similar as in the differential case, there are algorithms for finding closed form solutions of such recurrence, but we often prefer to represent a D-finite sequence by a recurrence of which it is a solution. Again quoting Salvy [77], about "25% of the sequences in Sloane's encyclopedia [82]" were D-finite in 2005. While the number of sequences recorded in this database has significantly increased since then, according to Yurkevich [92], the percentage of D-finite entries remained more or less stable, which means that there currently are about 100000 cases.

The word "D-finite" was proposed by Stanley in his 1980 survey paper [83], where he summarized a number of useful properties of D-finite functions that were already known in the 19th century. For example, it is easy to see that a formal power series is D-finite (in the differential sense) if and only if its coefficient sequence is D-finite (in the discrete sense). Moreover, D-finiteness is preserved under addition, multiplication, and various other operations.

The notion of D-finiteness becomes more interesting and more powerful in the case of several variables. Here we say that a function in several variables is D-finite if it is a solution of a system of partial linear differential (or recurrence) equations with polynomial coefficients which is such that its solution space has only finite dimension. If instead of equations we talk about operators that act on functions, the condition amounts to saying that the left ideal containing all the operators which map the function to zero should have dimension zero. Yet another way to say the same thing is that for each variable the function should satisfy a linear differential equation or a linear recurrence which contains only derivations or shifts with respect to the chosen variable and polynomial coefficients (that may contain all variables).

Computational aspects of D-finite functions have been studied since the early 1990s, and continue to be an active research area. Every year, the program of ISSAC includes papers which in one way or another provide some new algorithmic insight to D-finiteness. This work is not only of theoretical interest, but has also led to implementations in software packages, and these software packages are nowadays routinely applied in so many different contexts that it is difficult to give a reasonably complete overview. Here is just

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a list of some example application areas (adapted from Sect. 1.4 of [56]), along with some sample references:

- Enumeration of lattice walks [12, 17, 26],
- Permutation patterns [11, 18],
- Determinant evaluations [38, 65, 96],
- Graph counting [33, 44, 73],
- Analysis of algorithms [39],
- Program verification [49, 66, 67],
- Statistical mechanics [13, 24, 25],
- Particle physics [80],
- Special functions [7, 43],
- Analytic number theory [10, 97],
- Arithmetic number theory [30, 37, 48, 84],
- Experimental mathematics [46, 61, 88],
- Numerical engineering [8, 9, 74],
- Probability theory [6, 39],
- Knot theory [40, 41],
- Computational algebra [35],
- Biology [20, 70, 91],
- Coding theory [21–23],
- Control theory [76],
- Cryptography [27],
- Statistics [72],
- Spaceflight [52, 81],
- Sociology [42],
- Simulation [89].

Altogether, the development of algorithms for D-finite functions is a true success story for computer algebra.

2 ALGORITHMS

A univariate D-finite sequence is uniquely determined by a recurrence it satisfies and a suitable (finite!) number of sequence terms. Likewise, a univariate D-finite power series is uniquely determined by a differential equation and a suitable (finite!) number of series terms. Also in the multivariate case, D-finite objects are uniquely determined by some finitely many equations they satisfy, together with some finitely many initial values. This gives rise to a natural data structure for representing D-finite objects on a computer. Algorithms for D-finite functions operate primarily with these representations, but also with various other types of objects, such as truncated series, closed form expressions, etc. Altogether, there is a whole *computational ecosystem* around D-finite objects, and we are in the fortunate situations that this ecosystem comes with a lot of efficient algorithms for solving typical problems related to D-finite functions. In particular, and among other things, we know

- how to recover annihilating operators from sequence terms or series coefficients [47, 53, 56, 78],
- how to execute closure properties efficiently [15, 16, 55],
- how to solve symbolic summation and integration problems [31, 32, 34, 64, 94],
- how to find closed form solutions of differential and recurrence equations [56, 79, 87],
- how to extract residues, diagonals, or positive parts [17, 69],
- how to uncouple coupled systems of equations [1, 14, 98],
- how to test whether or not a given D-finite function is in fact algebraic [19],
- how to remove apparent singularities from linear operators [2, 3, 29, 51],
- how to expand D-finite objects as asymptotic series [39, 54, 90],
- how to compute arbitrary precision evaluations of D-finite functions [71, 85, 86].

Some of these operations are quite simple. For example, given a differential equation satisfied by a D-finite function f, say of order r, and a differential equation satisfied by a D-finite function g, say of order s, it is not hard to construct a differential equation which has $h := f + g$ among its solutions. The key observation is that f and all its derivatives belong to the vector space

$$C(x)f + C(x)f' + \cdots + C(x)f^{(r-1)}$$

generated by $f, f', \dots, f^{(r-1)}$ over the rational function field $C(x)$ and that g and all its derivatives belong to the vector space

$$C(x)g + C(x)g' + \cdots + C(x)g^{(s-1)}.$$

The reason is simply that the given differential equations allow to rewrite any higher order derivatives in terms of lower order derivatives. But then the sum $h = f + g$ and all its derivatives belong to the vector space

$$\begin{aligned} &C(x)f + C(x)f' + \cdots + C(x)f^{(r-1)} \\ &+ C(x)g + C(x)g' + \cdots + C(x)g^{(s-1)}. \end{aligned}$$

Since the dimension of space is at most $r + s$, the elements

$$h, h', \dots, h^{(r+s)}$$

must be linearly dependent. The dependence is the desired equation.

Its coefficients can be easily computed by solving a linear system. An ansatz with undetermined coefficients and coefficient comparison lead to a linear system over $C(x)$ with $r + s + 1$ variables and $r + s$ equations, which necessarily has a nonzero solution.

A more advanced computational technique for D-finite functions is known as *creative telescoping*. It applies in the multivariate setting and is used for definite summation/integration and related operations. Let's say we have a bivariate D-finite function $f(x, y)$ and we want to compute its residue $F(y) = [x^{-1}]f(x, y)$. The idea is to somehow find an annihilating operator of the integrand f which is of the form

$$P(y, D_y) - D_x Q(x, y, D_x, D_y),$$

where D_x, D_y refer to the partial derivations with respect to x and y, respectively. Since P is required to be free of x and D_x, it commutes with the residue extraction, and since $D_x Q(x, y, D_x, D_y)(f)$ is a derivative, its residue is zero. We thus find

$$P(y, D_y)(F) = 0.$$

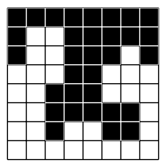
This is the desired relation for F.

The existence of an annihilating operator of the required form is ensured in the differential case, and under some technical assumptions also in the recurrence case. Starting in the 1990s [34, 93–95], the computation of such operators has been subject of intensive research throughout the years, with lots of papers on the subject appearing at ISSAC conferences, way too many to list them here. We refer to [28, 32, 56] and the references given there for details.

3 TWO EXAMPLES

Residue extraction may seem like a somewhat artificial operation. Is this really so important? In order to illustrate how residue extractions arise naturally in enumerative combinatorics, we will give two examples. They are chosen to be somewhat complementary to the context of lattice walk enumeration, for which the relevance of residue extractions has already been nicely presented by Bostan in his invited talk at ISSAC'21 [12].

3.1 The Gerrymander Sequence



Consider a rectangular grid of size $n\times m$ whose cells can be marked in black or white. Cells of the same color form connected regions, and we are interested in the number of ways to mark the cells of the array in such a way that there is exactly one black and exactly one white region. Moreover, the two regions should have exactly the same size. An example for $n = m = 8$ is given on the right.

Let us say we fix m and let n vary. Write $a_{n,k}$ for the number of arrays of size $n \times m$ with at most one region in each color and where the black region consists of exactly k cells, and let

$$a(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^k t^n$$

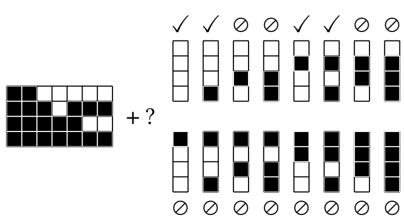
be the corresponding generating function. The number of arrays in which both regions have the same size is then $a_{n,nm/2}$ (understood to be zero if mn is odd). The corresponding generating function is

$$[x^{-1}]x^{-1}\,a(x^2,t/x^m) = [x^{-1}]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^{\overbrace{2k-nm-1}^{=-1\Leftrightarrow k=nm/2}}t^n,$$

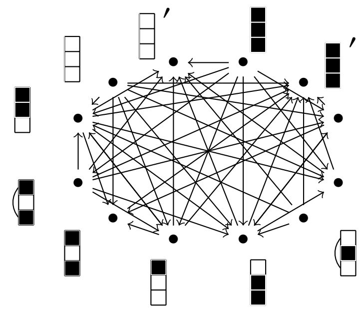
so if we can get hold of the bivariate series $a(x,t)$, then we are just a residue extraction away from the final result.

The bivariate series is not too hard to find, once we realize that it is equivalent to counting paths in a certain graph. Think of the arrays as being generated column by column, from left to right. Given an array, there are certain columns we are allowed to attach at the right hand side, and others which we cannot attach because they would violate the requirement of having at most two connected regions. The key observation is that in order to decide which columns are legitimate extensions, we do not need to know the full history. It is enough to know the previous column plus some information about whether or not separated cells of the same color belong to the same region.

Here is an example from our joint paper with Koutschan and Spahn [59]:



Now consider the directed graph whose vertices are all the possible situations that we may encounter in the right-most column of an array, i.e., the information which of its cells are marked in which color and whether they are connected through cells in earlier columns. Draw an edge from a vertex v_1 to a vertex v_2 if it is possible to turn a legal array whose right-most column is v_1 by adding a single column to a legal array whose right-most column is v_2. Here is what this graph looks like for $m = 3$:



The arcs indicate cells that have been connected in the past, and a prime indicates that the other color has already been seen in the past. Each state can also be followed by itself.

The arrays that we want to count amount to paths through this graph. The number of paths of length n from the ith to the jth vertex appears in the (i,j)th entry of the nth power of the adjacency matrix of the graph. Let us call this matrix A. If v_{start} and v_{end} are vectors which indicate legitimate starting and ending states, respectively (by having 1 at the positions corresponding to legitimate states and 0 in all other positions), then

$$v_{\text{end}}A^n v_{\text{start}}$$

is the number we are interested in. It follows that the generating function

$$\sum_{n=0}^{\infty}(v_{\text{end}}A^n v_{\text{start}})t^n = v_{\text{end}}\biggl(\sum_{n=0}^{\infty}A^n t^n\biggr)v_{\text{start}} = v_{\text{end}}(I-At)^{-1}v_{\text{start}}$$

is a rational function in t that we can easily compute.

This rational function is $a(1,t)$. It counts the total number arrays, regardless of how many black cells there are. To keep track of the number of black cells, it suffices to multiply each entry of A by the monomial x^k where k is the number black cells that gets added by attaching the respective column to the array. This turns A into an element of $\mathbb{Q}[x]$, and the calculation above yields $a(x,t)$ and shows that this is a rational function in x and t.

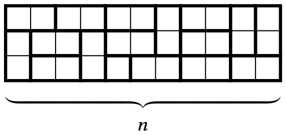
For example, for $m = 3$, we obtain

$$a(x,t) = \frac{\text{[[lengthy polynomial]]}}{(1-t)^2(1-tx^6)^2(1-tx^2)(1-tx^4)(tx^4+2tx^3+tx^2-1)},$$

and from here, via creative telescoping, a differential equation for $x^{-1}\,a(x^2,t/x^3)$. This case appeared as a Monthly problem a few years ago [75]. In our paper [59], we have worked out the case $m = 4$ along the same lines. For each specific choice of m, the problem can be solved in the same way, but the cost of computations grows quickly with m.

For square arrays, i.e., for $m = n$, the problem becomes much more difficult. The sequence obtained in this case is called the gerrymander sequence. It is known [45] that this sequence is not D-finite.

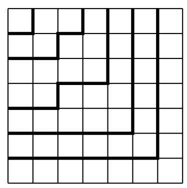
Homework: Let a_n be the number of ways to tile an array of size $3\times n$ with an arbitrary number of dimers and exactly one monomer, like this:



Show that (a_n) is D-finite and find a recurrence.

3.2 Hardinian Arrays

Another counting problem, which we considered together with Dougherty-Bliss [36], also led to some residue extraction, albeit in a somewhat different way. This problem came up when the method of "guessing with little data" proposed together with Koutschan at ISSAC'22 [57] was systematically applied to the OEIS [58, 82].



We are counting again ways to divide arrays into connected regions. The original specification of how this is done exactly is a bit cryptic, but it turns out to be equivalent to something that can be approached by a classical theorem from combinatorics. For simplicity, let us consider square arrays $n\times n$ only, and let $r\geq 1$ be fixed. Then the problem boils down to counting how many ways there are to draw $n-r-1$ paths starting from the left side of the array to the top, in such a way that no two paths intersect each other. An example for $n=7$ and $r=1$ is given on the right. We write $H_r(n)$ for the number of such tuples of paths.

For a single path from the uth row to the vth column, it is easy to see that there are $\binom{u+v}{u}$ options (if rows and columns are indexed starting from zero).

According to a theorem of Gessel and Viennot [68, Theorem 10.13.1], if we pick two sets A, B of n vertices in a graph and let $a_{i,j}$ be the number of paths from the ith vertex in A to the jth vertex in B, then the number of ways to pick paths from every vertex in A to some vertex in B such that no two paths intersect is exactly the determinant $\det((a_{i,j}))$.

Therefore, if we write $\Delta(n)_i^j$ for the determinant of the matrix obtained from $((\binom{u+v}{u}))_{u,v=0}^{n-1}$ by deleting the ith row and the jth column, then

$$H_1(n)=\sum_{i=0}^{n-2}\sum_{j=0}^{n-2}\Delta(n-1)_i^j.$$

There are several ways to show that this is equal to $\frac{1}{3}(4^{n-1}-1)$ for every $n\geq 1$, see [36] for four different proofs. In particular, the sequence is D-finite. It can further be shown that $\Delta(n)_i^j=\sum_{\ell=0}^{n-1}\binom{i}{\ell}\binom{j}{\ell}$.

For $r\geq 2$, we can argue in a similar way, but things become more messy, because there are more options for the start points and the end points. Anyhow, if for pairwise distinct $i_1,\dots,i_r$ and pairwise distinct $j_1,\dots,j_r$, we denote by $\Delta(n)_{i_1,\dots,i_r}^{j_1,\dots,j_r}$ the determinant of the matrix obtained from $((\binom{u+v}{u}))_{u,v=0}^{n-1}$ by deleting the rows $i_1,\dots,i_r$ and the columns $j_1,\dots,j_r$, then we have

$$H_r(n)=\sum_{\substack{0\leq i_1<\cdots<i_r<n-1\\ 0\leq j_1<\cdots<j_r<n-1}}\Delta(n-1)_{i_1,\dots,i_r}^{j_1,\dots,j_r}.$$

According to a theorem of Jacobi on determinants, we have

$$\Delta(n)_{i_1,\dots,i_r}^{j_1,\dots,j_r}=\begin{vmatrix}\Delta(n)_{i_1}^{j_1} & \cdots & \Delta(n)_{i_1}^{j_r}\\ \vdots & \ddots & \vdots\\ \Delta(n)_{i_r}^{j_1} & \cdots & \Delta(n)_{i_r}^{j_r}\end{vmatrix}.$$

For any fixed r, this means that $\Delta(n)_{i_1,\dots,i_r}^{j_1,\dots,j_r}$ is a certain polynomial expression of D-finite things, and therefore, by D-finite closure properties, again D-finite. As summation also preserves D-finiteness, we see that $H_r(n)$ is D-finite with respect to n for every fixed $r\geq 1$.

This proves in particular that $H_2(n)$ is D-finite, as conjectured in [58]. But the conjecture said more. Not only was it conjectured that a recurrence exists, but it was claimed that the sequence satisfies an very specific recurrence (explicitly stated in [58], but too lengthy to be reproduced here).

To prove that the conjectured recurrence for $r=2$ is correct, we need to compute a recurrence for

$$H_2(n)=\sum_{i_1,i_2}\sum_{j_1,j_2}\begin{vmatrix}\Delta(n-1)_{i_1}^{j_1} & \Delta(n-1)_{i_1}^{j_2}\\ \Delta(n-1)_{i_2}^{j_1} & \Delta(n-1)_{i_2}^{j_2}\end{vmatrix},$$

which, by expanding everything out, leads to two whopping six-fold sums:

$$\begin{aligned}S_1(n)&=\sum_{i_1\geq 0}\sum_{i_2>i_1}\sum_{j_1\geq 0}\sum_{j_2>j_1}\sum_{u=0}^{n}\sum_{v=0}^{n}\binom{u}{i_1}\binom{u}{j_1}\binom{v}{i_2}\binom{v}{j_2}\\ &=\sum_{u=0}^{n}\sum_{v=0}^{n}\underbrace{\Bigg(\sum_{i_1\geq 0}\sum_{i_2>i_1}\binom{u}{i_1}\binom{v}{i_2}\Bigg)}_{=:s(u,v)}\underbrace{\Bigg(\sum_{j_1\geq 0}\sum_{j_2>j_1}\binom{u}{j_1}\binom{v}{j_2}\Bigg)}_{=s(u,v)}\quad\text{and}\\ S_2(n)&=\sum_{i_1\geq 0}\sum_{i_2>i_1}\sum_{j_1\geq 0}\sum_{j_2>j_1}\sum_{u=0}^{n}\sum_{v=0}^{n}\binom{u}{i_1}\binom{u}{j_2}\binom{v}{i_2}\binom{v}{j_1}\\ &=\sum_{u=0}^{n}\sum_{v=0}^{n}\underbrace{\Bigg(\sum_{i_1\geq 0}\sum_{i_2>i_1}\binom{u}{i_1}\binom{v}{i_2}\Bigg)}_{=s(u,v)}\underbrace{\Bigg(\sum_{j_1\geq 0}\sum_{j_2>j_1}\binom{v}{j_1}\binom{u}{j_2}\Bigg)}_{=s(v,u)}.\end{aligned}$$

This is discouraging at first glance, but things become a bit nicer if we rephrase them in terms of generating functions. Noting that

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}s(u,v)x^uy^v=\frac{y}{(1-x-y)(1-2y)},$$

the generating functions for $s(u,v)^2$ and $s(u,v)s(v,u)$ can be expressed as Hadamard products:

$$\frac{y}{(1-x-y)(1-2y)}\odot\frac{y}{(1-x-y)(1-2y)},$$
$$\frac{y}{(1-x-y)(1-2y)}\odot\frac{x}{(1-x-y)(1-2x)}.$$

Hadamard products in turn can be rephrased as residues, because $a(x)\odot b(x)=[y^{-1}]\frac{1}{y}a(x)b(y)$ for any two series a, b, so these can be computed with creative telescoping.

Next, summing u from 0 to n and v from 0 to m amount to multiplying the resulting series by $\frac{1}{(1-x)(1-y)}$, which is not a big deal. But then we have to pick out from these bivariate series the terms $x^n y^m$ where n and m are equal. This can again be rephrased as a residue computation, because the diagonal of a bivariate series $a(x, y)$ is equal to $[y^{-1}]\frac{1}{y}a(x)b(y/x)$.

Using Koutschan's Mathematica implementation [62, 63], we were able to carry out these computations and confirm the conjectured recurrence.

Homework: Show that $\Delta(n) := \det((\binom{u+v}{v}))_{u,v=0}^{n} = 1$ for all n.

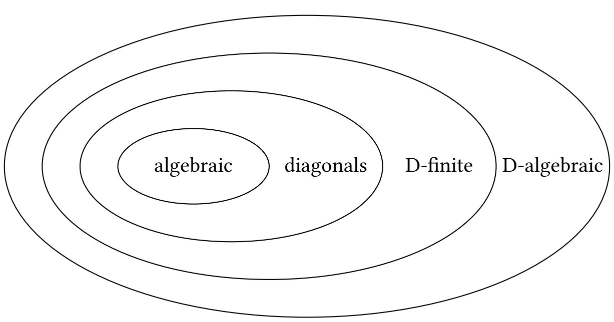
4 WHAT'S NEXT?

Thanks to the joint effort of many contributors, the computational ecosystem for D-finite functions has evolved to a robust and reliable machinery with can easily solve lots of problems that years ago would have been considered intractable. The topic of D-finiteness still provides some interesting research questions. For example, the detection of positivity of D-finite sequences is not sufficiently understood. See [50] for some recent progress on this matter. Still, for the main problems, we have satisfactory algorithmic solutions. It is time to move forward.

We can go in two directions: towards smaller classes of functions, with the hope that more efficient algorithms become available, or towards larger classes of functions, with the hope to cover certain functions that fail to be D-finite. Both directions are worthwhile, and we believe that in the coming years, we will see some interesting developments in each direction.

Going towards smaller classes, a class of functions that is not fully understood is the class of power series that can be written as the diagonal of a rational power series in several variables. It is known that all these series are D-finite, and but not every D-finite series belongs to this class. On the other hand, every algebraic function can be viewed as the diagonal of a rational function, but not every diagonal of a rational function is algebraic. The diagonals therefore form a proper intermediate class between algebraic and D-finite functions.

Going towards larger classes, a class that deserves more attention is the class of D-algebraic functions. These are functions f for which there is a nonzero polynomial in several variables such that $p(f, f', f'', \dots) = 0$. Clearly, every D-finite function is D-algebraic but not vice versa. Various applications naturally lead to D-algebraic functions that are not D-finite. There are also algorithms for certain tasks (cf. [5, 60] and the references given there), but it seems that more work is needed for this class.



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