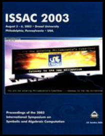
D-FINITENESS: A SUCCESS STORY

Manuel Kauers · Institute for Algebra · JKU



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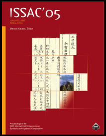

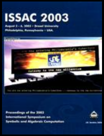

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- A number α is called algebraic if there are integers $c_0, \ldots, c_d$, not all zero, such that

$$c_0 + c_1\alpha + \cdots + c_d\alpha^d = 0.$$

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- A function f is called algebraic if there are polynomials $c_0, \dots, c_d$, not all zero, such that

$$c_0(x) + c_1(x)f(x) + \cdots + c_d(x)f(x)^d = 0.$$

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- A function f is called D-finite if there are polynomials $p_0, \ldots, p_r$, not all zero, such that

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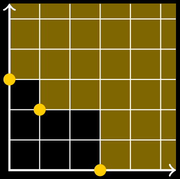

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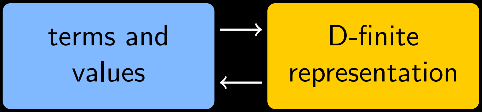
- $\frac{1}{n+1}\binom{2n}{n}$ — $(4n+2)a_n-(n+2)a_{n+1}=0$, $a_0=1$
- x^n — $f_{n+1}(x)-x\,f_n(x)=0$, $xf_n'(x)-nf_n(x)=0$, $f_0(x)=1$

A computational ecosystem

D-finite
representation

A computational ecosystem



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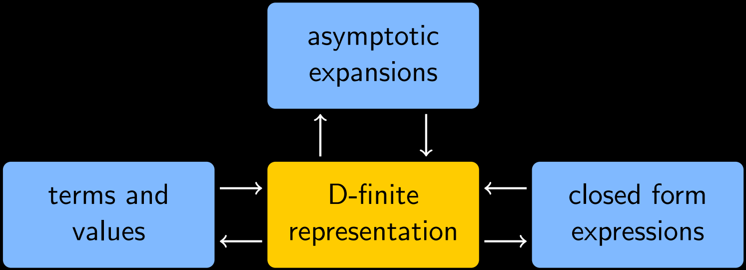

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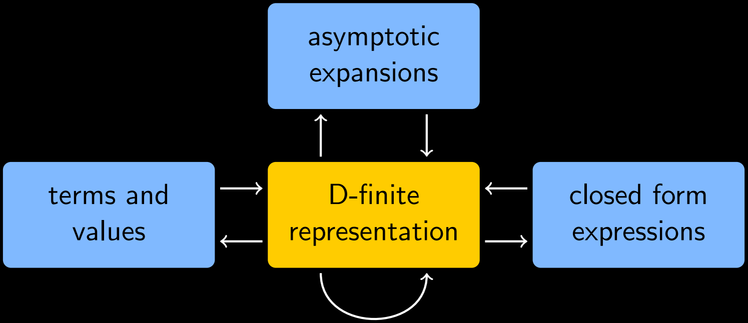
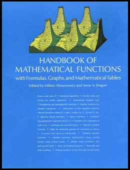


Who cares?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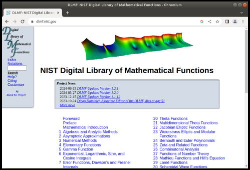
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21 Multidimensional Theta Functions
22 Jacobian Elliptic Functions
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28 Mathieu Functions and Hill's Equation
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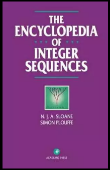
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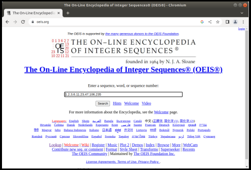
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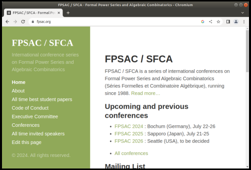
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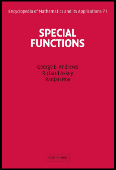
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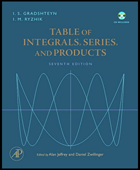
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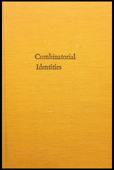
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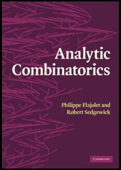
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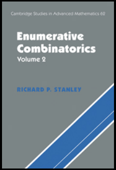
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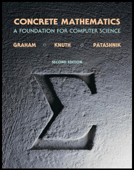
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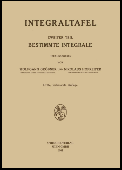
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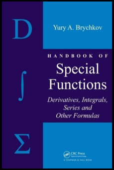
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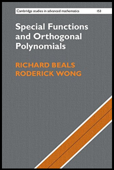
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Nico M. Temme

- Enumeration of lattice walks
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- Determinant evaluations
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- Arithmetic number theory
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- Probability theory
- Knot theory
- Computational algebra
- Biology
- Coding theory
- Control theory
- Cryptography
- Statistics
- Spaceflight
- Sociology
- Simulation

What can we do?

Introduction

- Functions, sequences, and series
- D-Finiteness
- Applications
- Computer Algebra
- Guessing
- Hermite-Pade Approximation

The Recurrence Case in One Variable

- Evaluation
- The Solution Space
- Closure Properties
- Generalized Series Solutions
- Polynomial and Rational Solutions
- Hypergeometric and D'Alembertian Solutions

The Differential Case in One Variable

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Operators

- Ore Algebras and Ore Actions
- Common right divisors and left multiples
- Several functions
- Factorization
- Several variables
- Gröbner bases

Summation and Integration

- The indefinite problem
- The definite problem
- Further closure properties
- Creative telescoping
- Bounds
- Reduction-based algorithms

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$$D_x\, f(x,y) = \sum_{n,k\in\mathbb{Z}} k\, a_{n,k}\, x^{k-1} t^n$$

$$
\begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & a_{-1,2} & a_{0,2} & a_{1,2} & \cdots \\
\cdots & a_{-1,1} & a_{0,1} & a_{1,1} & \cdots \\
\cdots & a_{-1,0} & a_{0,0} & a_{1,0} & \cdots \\
\cdots & a_{-1,-1} & a_{0,-1} & a_{1,-1} & \cdots \\
\cdots & a_{-1,-2} & a_{0,-2} & a_{1,-2} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

$$\begin{matrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & a_{-1,2} & a_{0,2} & a_{1,2} & \cdots \\ \cdots & a_{-1,1} & a_{0,1} & a_{1,1} & \cdots \\ \cdots & a_{-1,0} & a_{0,0} & a_{1,0} & \cdots \\ \cdots & a_{-1,-1} & a_{0,-1} & a_{1,-1} & \cdots \\ \cdots & a_{-1,-2} & a_{0,-2} & a_{1,-2} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

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\iddots & \vdots & \vdots & \vdots & \ddots
\end{matrix}
$$

$$
\begin{matrix}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & 3\,a_{-1,3} & 3\,a_{0,3} & 3\,a_{1,3} & \cdots \\
\cdots & 2\,a_{-1,2} & 2\,a_{0,2} & 2\,a_{1,2} & \cdots \\
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"telescoper" ↗ P **"certificate"** ↖ Q

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In particular, $[x^{-1}]f$ is D-finite.

There are tons of ISSAC papers on how to compute such P, Q.

Do we really need this?

Example 1

Computer Algebra in the Service of Enumerative Combinatorics

Alin Bostan
Inria, Université Paris-Saclay, France
alin.bostan@inria.fr

ABSTRACT

Classifying lattice walks in restricted lattices is an important problem in enumerative combinatorics. Recently, computer algebra has been used to explore and to solve a number of difficult questions related to lattice walks. We give an overview of recent results on structural properties (e.g., algebraicity versus transcendence) and on explicit formulas for generating functions of walks with small steps in the quarter plane. In doing so, we emphasize the algorithmic nature of the methodology, especially two important paradigms: "guess-and-prove" and "creative telescoping".

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms.**

KEYWORDS

Computer algebra; Experimental mathematics; Guess-and-Prove; Creative telescoping; Enumerative combinatorics; Lattice paths; Generating functions; D-finite functions; Algebraic functions.

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1 GENERAL PRESENTATION

1.1 Prelude

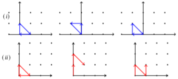
Consider the following innocent-looking problem.

A *tandem-walk* is a path in $\mathbb{Z}^2$ taking steps from $\{\uparrow, \leftarrow, \searrow\}$ only. Show that, for any integer $n \geq 0$, the following quantities are equal:

(i) the number a_n of tandem-walks of length n (i.e., using n steps), confined to the upper half-plane $\mathbb{Z} \times \mathbb{N}$, that start and end at $(0, 0)$;

(ii) the number b_n of tandem-walks of length n confined to the quarter plane $\mathbb{N}^2$, that start at $(0, 0)$ and finish on the diagonal $x = y$.

For instance, when $n = 3$, this common value is $a_3 = b_3 = 3$, as shown in the next picture, obtained by exhaustive enumeration.



It appears that this problem is far from being trivial. Several solutions exist, but none of them is elementary. One of the main aims of the present text is to convince the reader that this problem (and many others with a similar flavor) can be solved with the help of a computer. More precisely, computer algebra tools can be used to discover and to prove the following equalities

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!} \quad \text{and} \quad a_m = b_m = 0 \quad \text{if } 3 \nmid m. \qquad (1)$$

It goes without saying that such a simple and beautiful expression cannot be the result of mere chance. It turns out that closed forms are quite rare for this kind of enumeration problem. Nevertheless, even in the absence of nice formulas, the structural properties of the corresponding enumeration sequences reflect the symmetries of the step set and of the evolution domain. Equation (1) shows that the sequences (a_n) and (b_n) are P-recursive, that is, they satisfy a linear recurrence with polynomial coefficients (in the index n). Equivalently, their generating functions $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} b_n t^n$ are D-finite, that is, they satisfy linear differential equations with polynomial coefficients (in the variable t). On the methodological (i.e., computer-algebraic) side, one of the main messages that will emerge from the text is that, in the absence of closed formulas, the (recurrence/differential) equations themselves constitute the appropriate data structure to represent and manipulate P-recursive sequences and D-finite functions [65]. On the application (i.e., combinatorial) side, the main message is that these important properties of the enumeration sequences are intimately related to the finiteness of a certain group, naturally attached to the step set $\{\uparrow, \leftarrow, \searrow\}$.

1.2 General context: walks confined to cones

Example 2

Journal of Integer Sequences, Vol. 25 (2022),
Article 22.9.7

How Does the Gerrymander Sequence Continue?

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Abstract

We compute a few additional terms of the gerrymander sequence (OEIS sequence A348456) and provide guessed equations for the generating functions of some sequences in its context.

Computational challenge for you (and your computer)

1 message

Doron Zeilberger <doronzeil@gmail.com> Wed, May 4, 2022 at 3:43 PM
To: Manuel Kauers <manuel@kauers.de>
Cc: George Spahn <gs828@math.rutgers.edu>, Neil Sloane <njasloane@gmail.com>

Dear Manuel,

Hope you, Martina, and epsilon are doing well!

Can you (and your computers) meet the following challenge, in the secret url:https://sites.math.rutgers.edu/~zeilberg/EM22/C27.pdf

https://sites.math.rutgers.edu/~zeilberg/ChessChallenge.txt

If you do, I pledge to donate $100 to the OEIS in your honor.
Also, if you can do it systematically, this may lead to a joint paper with my student who can do other boards.

Best wishes,
Doron

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Gerrymandering (2), cont.

T(k, d) = no. of ways to dissect a k X k square board into d rook-connected regions of size k^2 / d.

(A348452, A348456, A172477, A004003)

$k \backslash d$	1	2	3	4	5	6	7	8	9	...	
1	1										
2	1	2	0	1							
3	1	0	10^a	0	0	0	0	0	1		
4	1	70^b	0	117	0	0	0	36^c	0	...	$1_{@16}$
5	1	0	0	0	4096	0	0	0	0	...	$1_{@25}$
6	1	80518	264500	442791	0	451206	0	0	✓	...	$1_{@36}$
7	1	0	0	0	0	0	✓	0	0	...	$1_{@49}$
8	1	?	0	?	0	0	0	✓	0	...	$1_{@16}$

Most wanted: T(8,2) = no. of ways to cut chessboard into 2 rook-connected regions of area 32

Ignore colors of chessboard squares; rotations, reflections count as different; regions need not have same shape.

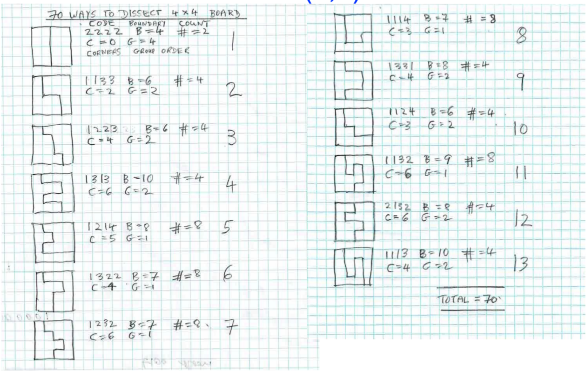
How large will T(8,2) be, roughly? How would you program it? How would you parallelize it?

Paul Zimmermann et al. in 2020 solved one of the RSA Challenge Problem,
It took them 2700 core years. How does T(8,2) compare?

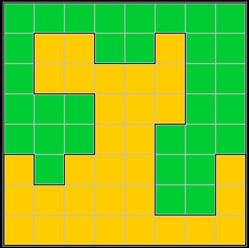
Gerrymandering (2), cont. T(4,2) = 70:

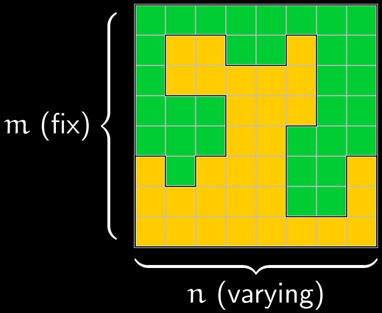
23 of 37 Math640.04.2022 C27.pdf 34.6%

Gerrymandering (2), cont. T(4,2) = 70:



Gerrymandering (2), cont. Tiling a Square with




m (fix)
n (varying)

Want:

$$a_n = \boxed{\begin{array}{c}\text{number of ways to split an } m \times n \text{ board}\\ \text{into two connected regions}\\ \text{of exactly the same size}\end{array}}$$

Want: $a(t) = \sum_{n=0}^{\infty} a_n t^n$ where

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Detour:

$$a_{n,k} = \boxed{\begin{array}{c}\text{number of ways to split an } m \times n \text{ board} \\ \text{into (at most) two connected regions,} \\ \text{with exactly } k \text{ yellow cells}\end{array}}$$

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Detour: $a(x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{n,k} x^k t^n$ where

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Key observation:

$$a(t) = [x^{-1}]\frac{1}{x}a\Big(x^2, \frac{t}{x^m}\Big)$$

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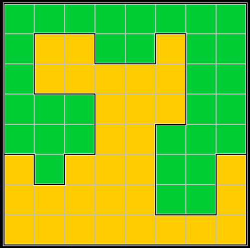
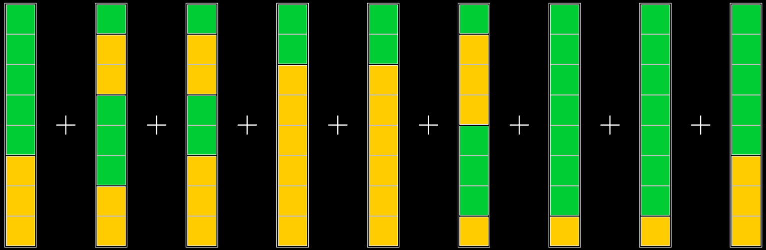
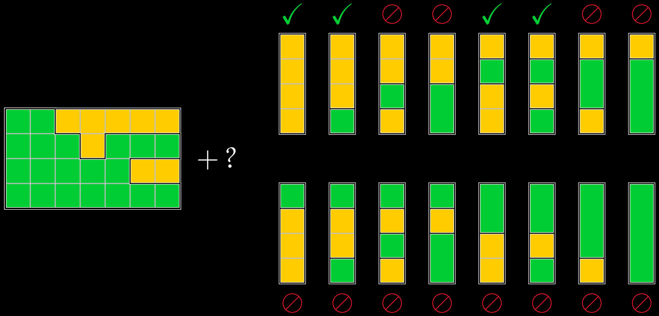
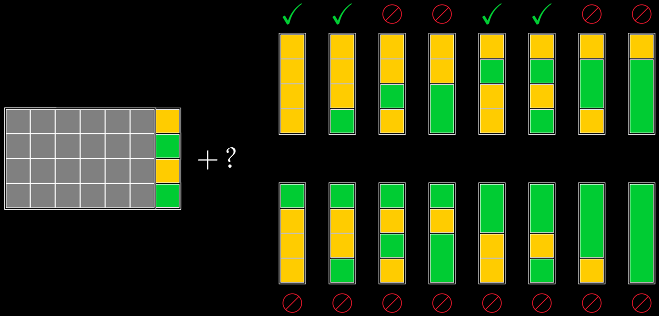
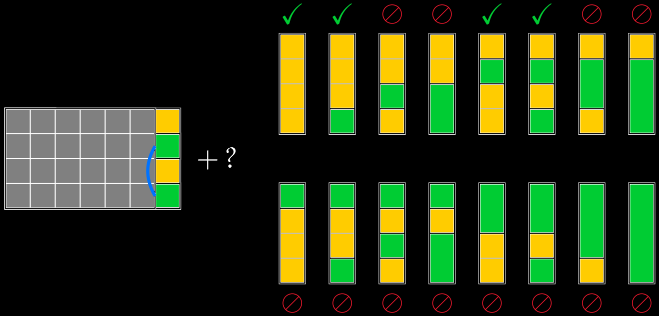
So we are done if we can find $a(x, t)$.

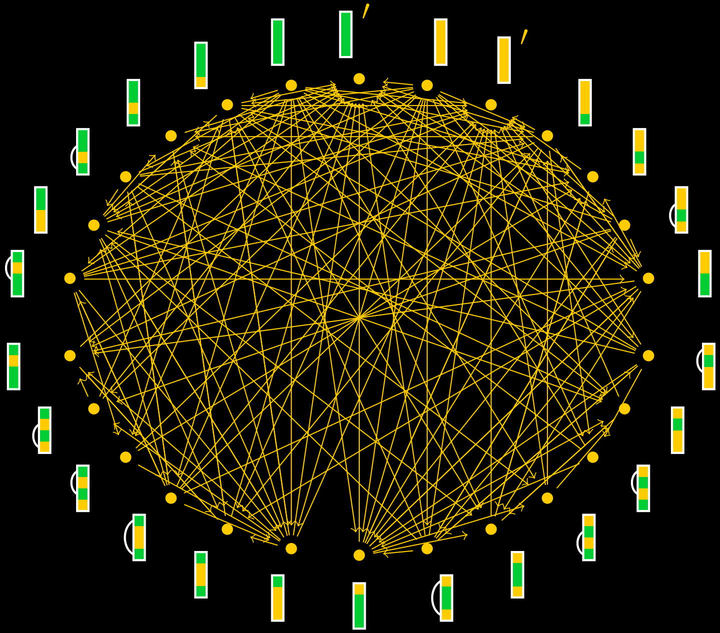












Let A be the adjacency matrix of this graph.

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The (i, j)-entry of A^n is the number of paths of length n from the ith to the jth vertex.

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This is $a(1, t)$. It is a rational function in t.

Each edge in the graph corresponds to adding a certain column to the array.

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Each such column has a specific number of yellow cells.

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Then $a(x,t) = v_{\text{init}}(1 - At)^{-1} v_{\text{final}}$.

This is a rational function in x and t.

$$
A = \begin{pmatrix}
x^4 & x^3 & x^3 & 0 & x^2 & x^3 & 0 & x^2 & 0 & x^2 & 0 & x & x^3 & 0 & x^2 & 0 & x^2 & 0 & x & x^2 & 0 & x & z & 0 & 0 & x^0 \\
0 & x^3 & 0 & 0 & x^2 & 0 & 0 & x^2 & 0 & 0 & 0 & x & 0 & 0 & x^2 & 0 & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x^4 & x^0 \\
0 & 0 & x^3 & 0 & x^2 & 0 & 0 & 0 & 0 & x^2 & 0 & x & 0 & 0 & 0 & 0 & x^2 & 0 & x & 0 & 0 & 0 & x & 0 & x^4 & x^0 \\
0 & 0 & 0 & x^3 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^4 & 0 \\
0 & x^3 & 0 & x^3 & x^2 & 0 & 0 & 0 & 0 & 0 & x^2 & x & 0 & 0 & x^2 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & x^4 & x^0 \\
0 & 0 & 0 & 0 & 0 & x^3 & 0 & x^2 & 0 & x^2 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & x & x & 0 & x^4 & x^0 \\
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0 & x^3 & 0 & 0 & 0 & 0 & x^3 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^4 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 & 0 & x^2 & 0 & x^2 & 0 & x & x^2 & 0 & x & x & 0 & x^4 & x^0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & x & 0 & 0 & x & 0 & 0 & 0 & x^0 \\
0 & 0 & 0 & x^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & x & 0 & 0 & 0 & x & 0 & 0 & x^0 \\
0 & x^3 & 0 & x^3 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 & x^2 & 0 & x^2 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & x^4 & x^0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & x^0 \\
0 & 0 & 0 & 0 & 0 & 0 & x^3 & 0 & 0 & 0 & x^2 & 0 & x^3 & 0 & x^2 & 0 & 0 & 0 & 0 & x^2 & 0 & x & x & 0 & x^4 & x^0 \\
0 & x^3 & 0 & 0 & 0 & 0 & x^3 & 0 & x^2 & 0 & 0 & 0 & x^3 & x^2 & 0 & 0 & 0 & 0 & 0 & x^2 & x & 0 & 0 & 0 & x^4 & x^0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & x^0 \\
0 & 0 & 0 & x^3 & 0 & 0 & x^3 & 0 & 0 & 0 & x^2 & 0 & x^3 & 0 & 0 & x^2 & 0 & 0 & 0 & x^2 & 0 & 0 & x & 0 & x^4 & x^0 \\
0 & x^3 & 0 & x^3 & x^2 & 0 & x^3 & 0 & x^2 & 0 & x^2 & x & x^3 & x^2 & 0 & x^2 & 0 & x & 0 & x^2 & x & 0 & x & x^0 & x^4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^0
\end{pmatrix}
$$

We were able to construct the rational functions $a(x, t)$ for $m = 1, \ldots, 7$.

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For $m = 1, 2, 3, 4$, we were able to obtain $a(x)$ by performing $[x^{-1}]$ on $a(x, t)$.

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For $m = 1, 2, 3, 4$, we were able to obtain $a(x)$ by performing $[x^{-1}]$ on $a(x, t)$.

Challenge: Find a differential equation for $a(x)$ for some $m \geq 5$.

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The number of ways to dissect an $m \times n$ board into two connected regions of equal size:

	1	2	3	4	5	6	7	8
1	0	1	0	1	0	1	0	1
2	1	2	3	4	5	6	7	8
3	0	3	0	19	0	85	0	355
4	1	4	19	70	245	856	2967	10164
5	0	5	0	245	0	8171	0	277969
6	1	6	85	856	8171	80518	806423	8059419
7	0	7	0	2967	0	806423	0	240009288
8	1	8	355	10164	277969	8059419	240009288	7157114189

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	1	2	3	4	5	6	7	8
1	0	1	0	1	0	1	0	1
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3	0	3	0	19	0	85	0	355
4	1	4	19	70	245	856	2967	10164
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2	1	2	3	4	5	6	7	8
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4	1	4	19	70	245	856	2967	10164
5	0	5	0	245	0	8171	0	277969
6	1	6	85	856	8171	80518	806423	8059419
7	0	7	0	2967	0	806423	0	240009288
8	1	8	355	10164	277969	8059419	240009288	7157114189

Guttmann and Jensen showed that the diagonal is **not** D-finite.

Homework: Let

$$a_n = \boxed{\begin{array}{c}\text{number of tilings of a } 3 \times n \text{ board} \\ \text{with dimers and exactly one monomer}\end{array}}$$

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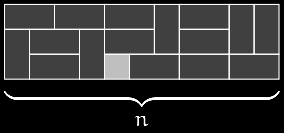

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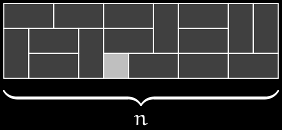

Show that (a_n) is D-finite and find a recurrence.

Example 3

Hardinian Arrays

Robert Dougherty-Bliss[a]   Manuel Kauers[b]

Submitted: Sep 1, 2023; Accepted: Mar 19, 2024; Published: Apr 5, 2024

Abstract

In 2014, R.H. Hardin contributed a family of sequences about king-moves on an array to the On-Line Encyclopedia of Integer Sequences (OEIS). The sequences were recently noticed in an automated search of the OEIS by Kauers and Koutschan, who conjectured a recurrence for one of them. We prove their conjecture as well as some older conjectures stated in the OEIS entries. We also have some new conjectures for the asymptotics of Hardin's sequences.

Mathematics Subject Classifications: 05A15, 33F10

1 Introduction

The On-Line Encyclopedia of Integer sequences [15] contains over 350,000 sequences and perhaps tens of thousands of conjectures about them. Here we resolve some of these conjectures related to a family of sequences due to R.H. Hardin.

For any positive integer r, let $H_r(n,k)$ be the number of $n \times k$ arrays which obey the following rules:

- The entry in position $(1,1)$ is 0, and the entry in position (n,k) is $\max(n,k) - r - 1$.
- The entry in position (i,j) must equal or be one more than each of the entries in positions $(i-1,j)$, $(i,j-1)$, and $(i-1,j-1)$.
- The entry in position (i,j) must be within r of $\max(i,j) - 1$.

We call such an arrangement of numbers a Hardinian array.

Equivalently, Hardinian arrays can be defined in terms of the king-distance between entries, i.e., the length of the shortest path that a king on a chessboard can take to get from one entry to the other. Using this notion, we can say that a Hardinian array has [illegible]

A253217 - OEIS

oeis.org/A253217

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A253217 Number of n X n nonnegative integer arrays with upper left 0 and lower right its king-move distance away minus 2 and every value within 2 of its king move distance from the upper left and every value increasing by 0 or 1 with every step right, diagonally se or down. 3

0, 0, 1, 19, 268, 3568, 47698, 649712, 9023385, 127419681, 1823918697, 26398702645, 385582981615, 5674890516295, 84060883775765, 1252066289632643, 18738613233957420, 281620474177057788, 4248088188086420832 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,4

COMMENTS Diagonal of A253223.

LINKS R. H. Hardin, Table of n, a(n) for n = 1..37

M. Kauers and C. Koutschan, Guessing with Little Data, ISSAC '22: Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation, July 2022, Pages 83-90.

M. Kauers and C. Koutschan, Some D-finite and some possibly D-finite sequences in the OEIS, arXiv:2303.02793 [cs.SC], 2023.

R. Dougherty-Bliss and M. Kauers, Hardinian Arrays, arXiv:2309.00487 [math.co], 2023.

FORMULA Recurrence: 32*(1 + n)*(1 + 2*n)^2*(161046 + 465785*n + 551943*n^2 +

Number of $n \times n$ nonnegative integer arrays with upper left 0 and lower right its king-move distance away minus 2 and every value within 2 of its king move distance from the upper left and every value increasing by 0 or 1 with every step right, diagonally SE or down.

$H_2(4,4) = 19$, because:

$$\begin{matrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&0&0&1\\0&0&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&0&0&1\\0&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&0&0&1\\1&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&0&1&1\\0&0&1&1\\1&1&1&1\end{matrix},$$

$$\begin{matrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&0&1&1\\1&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&1&1&1\\0&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\0&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&0&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix},$$

$$\begin{matrix}0&0&1&1\\0&0&1&1\\0&0&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&1&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&1&1\\0&0&1&1\\1&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&1&1\\0&1&1&1\\0&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&0&1&1\\0&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix},$$

$$\begin{matrix}0&0&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&1&1&1\\0&1&1&1\\0&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&1&1&1\\0&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix}, \quad \begin{matrix}0&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{matrix}.$$

$H_2(n, n)$ begins as follows:

$$0, 0, 1, 19, 268, 3568, 47698, 649712, 9023385, 127419681, \ldots$$

$$(201600n^9 + 4942080n^8 + 53078112n^7 + 327661728n^6 + 1280700480n^5 + 3285342016n^4 + 5528828352n^3 + 5883447104n^2 + 3591093120n + 957662208)a(n) + (-970200n^9 - 24199560n^8 - 264810744n^7 - 1667830872n^6 - 6659340648n^5 - 17470825688n^4 - 30096410912n^3 - 32804461872n^2 - 20514211488n - 5603970816)a(n+1) + (589050n^9 + 14827590n^8 + 163756656n^7 + 1040895564n^6 + 4194035058n^5 + 11101344742n^4 + 19289250308n^3 + 21198776056n^2 + 13360158000n + 3676219776)a(n+2) + (294525n^9 + 7319295n^8 + 79828578n^7 + 501335472n^6 + 1997003589n^5 + 5229549731n^4 + 8997110634n^3 + 9799013608n^2 + 6125859120n + 1673566848)a(n+3) + (-121275n^9 - 3053295n^8 - 33716268n^7 - 214212552n^6 - 862421763n^5 - 2280190003n^4 - 3956305720n^3 - 4340670060n^2 - 2730542400n - 749859264)a(n+4) + (6300n^9 + 163890n^8 + 1863666n^7 + 12150660n^6 + 50023284n^5 + 134779202n^4 + 237527338n^3 + 263895164n^2 + 167643648n + 46381248)a(n+5) \stackrel{?}{=} 0$$

Let's prove this.

Let's prove this.

But first, let's consider the case $r = 1$.

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But first, let's consider the case $r = 1$.

Or maybe let's even start with $r = 0$.

0	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

0	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

0	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

$H_0(n, k) = 1$ for all n, k.

0

5

$$H_1(n, n) = ?$$

0						5
						5
						5
						5
						5
						5
5	5	5	5	5	5	5

$$H_1(n, n) = ?$$

0	1	2	2	3	4	5
1						5
2						5
3						5
3						5
4						5
5	5	5	5	5	5	5

$$H_1(n, n) = ?$$

0	1	2	2	3	4	5
1						5
2						5
3						5
3						5
4						5
5	5	5	5	5	5	5

$$H_1(n, n) = ?$$

0	1	2	2	3	4	5
1	1	2	3	3	4	5
2	2	2	3	3	4	5
3	3	3	3	4	4	5
3	3	4	4	4	5	5
4	4	4	4	5	5	5
5	5	5	5	5	5	5

$$H_1(n, n) = ?$$

0	1	2	2	3	4	5
1	1	2	3	3	4	5
2	2	2	3	3	4	5
3	3	3	3	4	4	5
3	3	4	4	4	5	5
4	4	4	4	5	5	5
5	5	5	5	5	5	5

$$H_1(n, n) = ?$$

$H_1(n, n)$ is the number of non-crossing lattice path tuples $(P_1, \ldots, P_{n-1})$ where each path P_i starts somewhere on the left and ends somewhere at the top.

$H_r(n, n)$ is the number of non-crossing lattice path tuples $(P_1, \ldots, P_{n-r})$ where each path P_i starts somewhere on the left and ends somewhere at the top.

Theorem (Lindström-Gessel-Viennot). In a directed graph $G = (V, E)$, let

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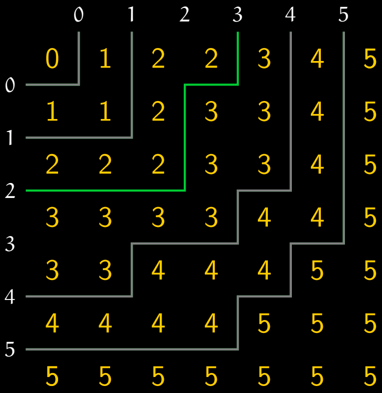
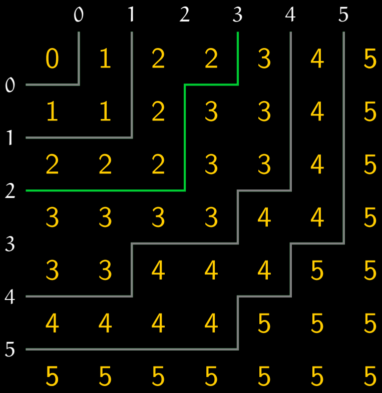
Then:

$$\boxed{\begin{array}{c}\text{number of non-crossing path tuples}\\ \text{from } (s_1, \ldots, s_{n-r}) \text{ to } (e_1, \ldots, e_{n-r})\end{array}} = \det((a_{i,j}))_{i,j=1}^{n-r}.$$

0	1	2	2	3	4	5
1	1	2	3	3	4	5
2	2	2	3	3	4	5
3	3	3	3	4	4	5
3	3	4	4	4	5	5
4	4	4	4	5	5	5
5	5	5	5	5	5	5

0	1	2	2	3	4	5
1	1	2	3	3	4	5
2	2	2	3	3	4	5
3	3	3	3	4	4	5
3	3	4	4	4	5	5
4	4	4	4	5	5	5
5	5	5	5	5	5	5


0 1 2 3 4 5
0 1 2 2 3 4 5
0
1 1 2 3 3 4 5
1
2 2 2 3 3 4 5
2
3 3 3 3 4 4 5
3
3 3 4 4 4 5 5
4
4 4 4 4 5 5 5
5
5 5 5 5 5 5 5



$$\boxed{\#\text{paths from } u \text{ to } v} = \binom{u+v}{v}$$

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

2 is not an
ending position

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

2 is not an ending position

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

3 is not a starting position

2 is not an
ending position

$$\Delta(5)^2_3 = \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

3 is not a
starting position

2 is not an ending position

3 is not a starting position

$$\Delta(5)_3^2 = \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

$$H_1(n,n) = \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} \Delta(n-1)_i^j.$$

Homework: Show that

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- $\det((\binom{u+v}{v}))_{u,v=0}^{n-1} = 1$

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Homework: Show that

- $\det((\binom{u+v}{v}))_{u,v=0}^{n-1} = 1$
- $\Delta(n)_i^j = \sum_{\ell=0}^{n-1} \binom{\ell}{i}\binom{\ell}{j}$
- $\sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \Delta(n-1)_i^j = \frac{1}{3}(4^{n-1}-1)$

0

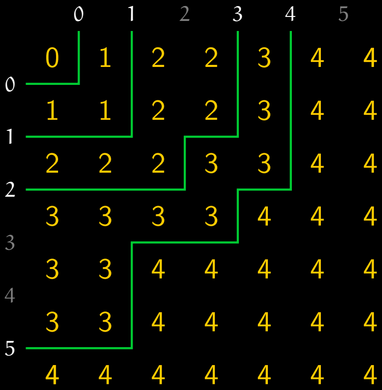
4

0						4
						4
						4
						4
						4
						4
4	4	4	4	4	4	4

0	1	2	2	3	4	4
1						4
2						4
3						4
3						4
3						4
4	4	4	4	4	4	4

0	1	2	2	3	4	4
1	1	2	2	3	4	4
2	2	2	3	3	4	4
3	3	3	3	4	4	4
3	3	4	4	4	4	4
3	3	4	4	4	4	4
4	4	4	4	4	4	4

0	1	2	2	3	4	4
1	1	2	2	3	4	4
2	2	2	3	3	4	4
3	3	3	3	4	4	4
3	3	4	4	4	4	4
3	3	4	4	4	4	4
4	4	4	4	4	4	4


0 1 2 3 4 5
0 1 2 2 3 4 4
0
1 1 2 2 3 4 4
1
2 2 2 3 3 4 4
2
3 3 3 3 4 4 4
3
3 3 4 4 4 4 4
4
3 3 4 4 4 4 4
5
4 4 4 4 4 4 4

2 is not an ending position

5 is not an ending position

$$\begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

3 is not a starting position

4 is not a starting position

2 is not an ending position

5 is not an ending position

$$\Delta(5)^{2,5}_{3,4} = \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

3 is not a starting position

4 is not a starting position

2 is not an ending position

5 is not an ending position

$$\Delta(5)^{2,5}_{3,4} = \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \binom{4}{0} & \binom{5}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \binom{4}{1} & \binom{5}{1} & \binom{6}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2} & \binom{6}{2} & \binom{7}{2} \\ \binom{3}{3} & \binom{4}{3} & \binom{5}{3} & \binom{6}{3} & \binom{7}{3} & \binom{8}{3} \\ \binom{4}{4} & \binom{5}{4} & \binom{6}{4} & \binom{7}{4} & \binom{8}{4} & \binom{9}{4} \\ \binom{5}{5} & \binom{6}{5} & \binom{7}{5} & \binom{8}{5} & \binom{9}{5} & \binom{10}{5} \end{vmatrix}$$

3 is not a starting position

4 is not a starting position

$$H_2(n,n) = \sum_{i_1=0}^{n-2} \sum_{i_2=i_1+1}^{n-2} \sum_{j_1=0}^{n-2} \sum_{j_2=j_1+1}^{n-2} \Delta(n-1)^{j_1,j_2}_{i_1,i_2}.$$

$$H_r(n,n) = \sum_{\substack{0 \le i_1 < \cdots < i_r < n-1 \\ 0 \le j_1 < \cdots < j_r < n-1}} \Delta(n-1)^{j_1,\ldots,j_r}_{i_1,\ldots,i_r}$$

$$H_r(n,n) = \sum_{\substack{0\le i_1<\cdots<i_r<n-1\\ 0\le j_1<\cdots<j_r<n-1}} \Delta(n-1)_{i_1,\ldots,i_r}^{j_1,\ldots,j_r}$$

Is this D-finite?

Jacobi's determinant identity implies

$$\Delta(n-1)^{j_1,\ldots,j_r}_{i_1,\ldots,i_r} = \begin{vmatrix} \Delta(n-1)^{j_1}_{i_1} & \cdots & \Delta(n-1)^{j_r}_{i_1} \\ \vdots & \ddots & \vdots \\ \Delta(n-1)^{j_1}_{i_r} & \cdots & \Delta(n-1)^{j_r}_{i_r} \end{vmatrix}$$

Jacobi's determinant identity implies

$$\Delta(n-1)^{j_1,\dots,j_r}_{i_1,\dots,i_r} = \begin{vmatrix} \Delta(n-1)^{j_1}_{i_1} & \cdots & \Delta(n-1)^{j_r}_{i_1} \\ \vdots & \ddots & \vdots \\ \Delta(n-1)^{j_1}_{i_r} & \cdots & \Delta(n-1)^{j_r}_{i_r} \end{vmatrix}$$

This is D-finite in n for every fixed r and $i_1,\dots,i_r,j_1,\dots,j_r$.

Jacobi's determinant identity implies

$$\Delta(n-1)^{j_1,\dots,j_r}_{i_1,\dots,i_r} = \begin{vmatrix} \Delta(n-1)^{j_1}_{i_1} & \cdots & \Delta(n-1)^{j_r}_{i_1} \\ \vdots & \ddots & \vdots \\ \Delta(n-1)^{j_1}_{i_r} & \cdots & \Delta(n-1)^{j_r}_{i_r} \end{vmatrix}$$

This is D-finite in n for every fixed r and $i_1, \dots, i_r, j_1, \dots, j_r$.

Therefore, $H_r(n,n) = \sum_{i_1,\dots,i_r} \sum_{j_1,\dots,j_r} \Delta(n-1)^{j_1,\dots,j_r}_{i_1,\dots,i_r}$ is D-finite.

$$
\begin{gathered}
(201600n^9 + 4942080n^8 + 53078112n^7 + 327661728n^6 + \\
1280700480n^5 + 3285342016n^4 + 5528828352n^3 + 5883447104n^2 + \\
3591093120n + 957662208)a(n) + (-970200n^9 - 24199560n^8 - \\
264810744n^7 - 1667830872n^6 - 6659340648n^5 - 17470825688n^4 - \\
30096410912n^3 - 32804461872n^2 - 20514211488n - \\
5603970816)a(n+1) + (589050n^9 + 14827590n^8 + 163756656n^7 + \\
1040895564n^6 + 4194035058n^5 + 11101344742n^4 + 19289250308n^3 + \\
21198776056n^2 + 13360158000n + 3676219776)a(n+2) + (294525n^9 + \\
7319295n^8 + 79828578n^7 + 501335472n^6 + 1997003589n^5 + \\
5229549731n^4 + 8997110634n^3 + 9799013608n^2 + 6125859120n + \\
1673566848)a(n+3) + (-121275n^9 - 3053295n^8 - 33716268n^7 - \\
214212552n^6 - 862421763n^5 - 2280190003n^4 - 3956305720n^3 - \\
4340670060n^2 - 2730542400n - 749859264)a(n+4) + (6300n^9 + \\
163890n^8 + 1863666n^7 + 12150660n^6 + 50023284n^5 + 134779202n^4 + \\
237527338n^3 + 263895164n^2 + 167643648n + 46381248)a(n+5) \stackrel{?}{=} 0
\end{gathered}
$$

$$H_2(n,n) = \sum_{i_1=0}^{n-2} \sum_{i_2=i_1+1}^{n-2} \sum_{j_1=0}^{n-2} \sum_{j_2=j_1+1}^{n-2} \Delta(n-1)_{i_1,i_2}^{j_1,j_2}$$

$$H_2(n,n) = \sum_{i_1=0}^{n-2} \sum_{i_2=i_1+1}^{n-2} \sum_{j_1=0}^{n-2} \sum_{j_2=j_1+1}^{n-2} \begin{vmatrix} \Delta(n-1)_{i_1}^{j_1} & \Delta(n-1)_{i_1}^{j_2} \\ \Delta(n-1)_{i_2}^{j_1} & \Delta(n-1)_{i_2}^{j_2} \end{vmatrix}$$

$$H_2(n,n) = \sum_{i_1=0}^{n-2} \sum_{i_2=i_1+1}^{n-2} \sum_{j_1=0}^{n-2} \sum_{j_2=j_1+1}^{n-2} \begin{vmatrix} \sum_{\ell=0}^{n-1} \binom{\ell}{i_1}\binom{\ell}{j_1} & \sum_{\ell=0}^{n-1} \binom{\ell}{i_1}\binom{\ell}{j_2} \\ \sum_{\ell=0}^{n-1} \binom{\ell}{i_2}\binom{\ell}{j_1} & \sum_{\ell=0}^{n-1} \binom{\ell}{i_2}\binom{\ell}{j_2} \end{vmatrix}$$

$H_2(n, n) = S_1(n) - S_2(n)$, where

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$$S_1(n) = \sum_{i_1 \geq 0} \sum_{i_2 > i_1} \sum_{j_1 \geq 0} \sum_{j_2 > j_1} \sum_{u=0}^{n} \sum_{v=0}^{n} \binom{u}{i_1} \binom{u}{j_1} \binom{v}{i_2} \binom{v}{j_2}$$

$$S_2(n) = \sum_{i_1 \geq 0} \sum_{i_2 > i_1} \sum_{j_1 \geq 0} \sum_{j_2 > j_1} \sum_{u=0}^{n} \sum_{v=0}^{n} \binom{u}{i_1} \binom{u}{j_2} \binom{v}{i_2} \binom{v}{j_1}$$

$H_2(n, n) = S_1(n) - S_2(n)$, where

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} \Big(\sum_{i_1 \geq 0} \sum_{i_2 > i_1} \binom{u}{i_1} \binom{v}{i_2} \Big) \Big(\sum_{j_1 \geq 0} \sum_{j_2 > j_1} \binom{u}{j_1} \binom{v}{j_2} \Big)$$

$$S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} \Big(\sum_{i_1 \geq 0} \sum_{i_2 > i_1} \binom{u}{i_1} \binom{v}{i_2} \Big) \Big(\sum_{j_1 \geq 0} \sum_{j_2 > j_1} \binom{v}{j_1} \binom{u}{j_2} \Big)$$

$H_2(n,n) = S_1(n) - S_2(n)$, where

$$S_1(n) = \sum_{u=0}^{n}\sum_{v=0}^{n}\underbrace{\left(\sum_{i_1\geq 0}\sum_{i_2>i_1}\binom{u}{i_1}\binom{v}{i_2}\right)}_{=:s(u,v)}\underbrace{\left(\sum_{j_1\geq 0}\sum_{j_2>j_1}\binom{u}{j_1}\binom{v}{j_2}\right)}_{=s(u,v)}$$

$$S_2(n) = \sum_{u=0}^{n}\sum_{v=0}^{n}\underbrace{\left(\sum_{i_1\geq 0}\sum_{i_2>i_1}\binom{u}{i_1}\binom{v}{i_2}\right)}_{=s(u,v)}\underbrace{\left(\sum_{j_1\geq 0}\sum_{j_2>j_1}\binom{v}{j_1}\binom{u}{j_2}\right)}_{=s(v,u)}$$

$$H_2(n, n) = S_1(n) - S_2(n), \text{ where}$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u, v)^2$$

$$S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u, v)s(v, u)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v = \frac{y}{(1-x-y)(1-2y)}$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$f(x,y) := \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v = \frac{y}{(1-x-y)(1-2y)}$$

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$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)^2\, x^u y^v = f(x,y) \odot_{x,y} f(x,y)$$

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)s(v,u)\, x^u y^v = f(x,y) \odot_{x,y} f(y,x)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$f(x,y) := \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v = \frac{y}{(1-x-y)(1-2y)}$$

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)^2\, x^u y^v = [X^{-1}][Y^{-1}] \tfrac{1}{XY} f(\tfrac{x}{X}, \tfrac{y}{Y}) f(X,Y)$$

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)s(v,u)\, x^u y^v = [X^{-1}][Y^{-1}] \tfrac{1}{XY} f(\tfrac{x}{X}, \tfrac{y}{Y}) f(Y,X)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

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$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)^2\, x^u y^v = g_1(x,y)$$

$$\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)s(v,u)\, x^u y^v = g_2(x,y)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$f(x,y) := \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v = \frac{y}{(1-x-y)(1-2y)}$$

$$\frac{1}{(1-x)(1-y)} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)^2\, x^u y^v = \frac{1}{(1-x)(1-y)} g_1(x,y)$$

$$\frac{1}{(1-x)(1-y)} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)s(v,u)\, x^u y^v = \frac{1}{(1-x)(1-y)} g_2(x,y)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

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$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{u=0}^{n} \sum_{v=0}^{m} s(u,v)s(v,u)\, x^n y^m = \frac{1}{(1-x)(1-y)} g_2(x,y)$$

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$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$f(x,y) := \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v = \frac{y}{(1-x-y)(1-2y)}$$

$$\sum_{n=0}^{\infty} \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2\, x^n = [y^{-1}] \frac{1}{y(1-\frac{x}{y})(1-y)} g_1(\frac{x}{y}, y)$$

$$\sum_{n=0}^{\infty} \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)\, x^n = [y^{-1}] \frac{1}{y(1-\frac{x}{y})(1-y)} g_2(\frac{x}{y}, y)$$

$$S_1(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)^2 \qquad S_2(n) = \sum_{u=0}^{n} \sum_{v=0}^{n} s(u,v)s(v,u)$$

$$f(x,y) := \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} s(u,v)\, x^u y^v = \frac{y}{(1-x-y)(1-2y)}$$

$$\sum_{n=0}^{\infty} S_1(n)\, x^n = [y^{-1}] \frac{1}{y(1-\frac{x}{y})(1-y)} g_1(\frac{x}{y}, y)$$

$$\sum_{n=0}^{\infty} S_2(n)\, x^n = [y^{-1}] \frac{1}{y(1-\frac{x}{y})(1-y)} g_2(\frac{x}{y}, y)$$

Altogether,

$$\sum_{n=0}^{\infty} H_2(n)x^n = [y^{-1}]\frac{1}{y(1-\frac{x}{y})(1-y)}\Big([X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(X,Y)$$
$$- [X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(Y,X)\Big)$$

Altogether,

$$\sum_{n=0}^{\infty} H_2(n)x^n = [y^{-1}]\frac{1}{y(1-\frac{x}{y})(1-y)}\Big([X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(X,Y) - [X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(Y,X)\Big)$$

diagonal

Hadamard product

Altogether,

$$\sum_{n=0}^{\infty} H_2(n)x^n = [y^{-1}]\frac{1}{y(1-\frac{x}{y})(1-y)}\Big([X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(X,Y) - [X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(Y,X)\Big)$$

↑ **diagonal**

↑ **Hadamard product**

From this expression we can compute a recurrence for $H_2(n)$.

Altogether,

$$\sum_{n=0}^{\infty} H_2(n)x^n = [y^{-1}]\frac{1}{y(1-\frac{x}{y})(1-y)}\Bigg([X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(X,Y) - [X^{-1}][Y^{-1}]\frac{1}{XY}f(\frac{x}{Xy},\frac{y}{Y})f(Y,X)\Bigg)$$

↑ **diagonal**

↑ **Hadamard product**

From this expression we can compute a recurrence for $H_2(n)$.

This proves the conjecture.

$$H_3(n,n) = \sum_{i_1=0}^{n-2} \sum_{i_2=i_1+1}^{n-2} \sum_{i_3=i_2+1}^{n-2} \sum_{j_1=0}^{n-2} \sum_{j_2=j_1+1}^{n-2} \sum_{j_3=j_2+1}^{n+2} \Delta(n-1)_{i_1,i_2,i_3}^{j_1,j_2,j_3}$$

$$H_3(n,n) = \sum_{i_1=0}^{n-2} \sum_{i_2=i_1+1}^{n-2} \sum_{i_3=i_2+1}^{n-2} \sum_{j_1=0}^{n-2} \sum_{j_2=j_1+1}^{n-2} \sum_{j_3=j_2+1}^{n+2} \Delta(n-1)_{i_1,i_2,i_3}^{j_1,j_2,j_3}$$

For this case, we only have a guessed recurrence.

$$-4096(3+n)(4+n)(5+2n)^2(7+2n)^2(9+2n)(391119595606371607976649348669505536 00+197256001218352660384779452476642017280n+47913909939183746025935297377296193 1264n^2+7464478832450605256030516869289587535 36n^3+8379365938889419251446971522625127914 88n^4+72192645487612025075027707989708437084 8n^5+496464743768781009302932567185965249872n^6+27982543031752069386687296126783941528n^7+13171397596185267410678393708867391528 0n^8+5248954829480587938246188512021557174 0n^9+178907322001050027024165740415934604 44n^{10}+525513658254670508797478207858958572 0n^{11}+1337667763819313718578568645595657352 n^{12}+296218993034590539963226189727327155 n^{13}+572081367648684325745 6901054 6 523521n^{14}+9647442709259967488980658747655612n^{15}+1420638077333654914542761127473109n^{16}+182455504359557324965045807480254n^{17}+20387670169703284105775663020443n^{18}+1974413313400097889736606343574n^{19}+16480692899220399010656228226 91n^{20}+117681554291358291980061721 86n^{21}+7115966230076414873841488 31n^{22}+3594319586461816552936728 0n^{23}+1488433854867162397844355n^{24}+492150869690724673600 50n^{25}+1249381284486939221625n^{26}+22863320510567748750n^{27}+268460266512178125n^{28}+1518930036421875n^{29})a(n)+2304(4+n)(7+2n)^2(9+2n)(4076853907052385086454156738 9110108160000+\ldots)a(1+n)-576(9+2n)(\ldots)a(2+n)+48(\ldots)a(3+n)+36(\ldots)a(4+n)-18(\ldots)a(5+n)+3(\ldots)a(6+n)+18(7+n)(\ldots)a(7+n)-9(7+n)(8+n)^2(17+2n)(\ldots)a(8+n)+2(7+n)(8+n)^2(9+n)^2(17+2n)(19+2n)(\ldots)a(9+n)\stackrel{?}{=}0$$


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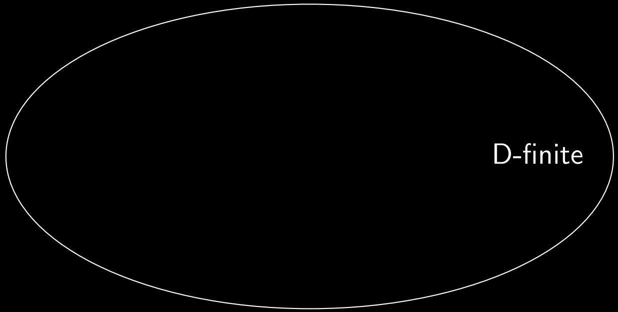
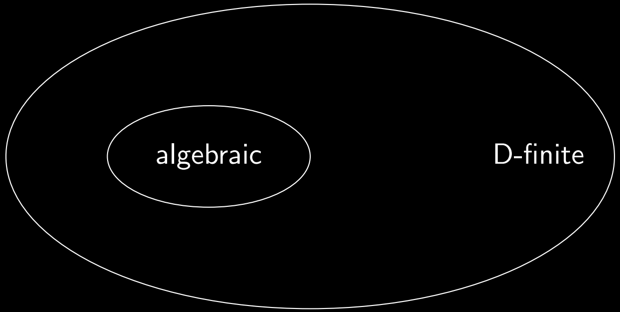
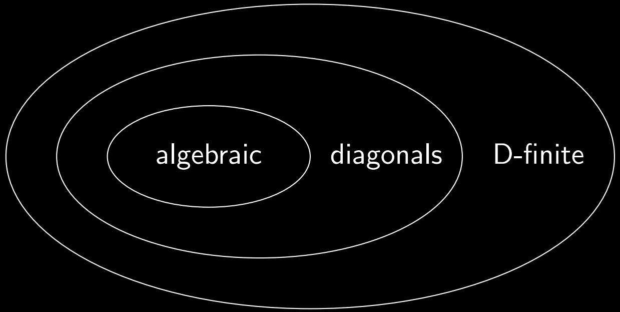
What's next?

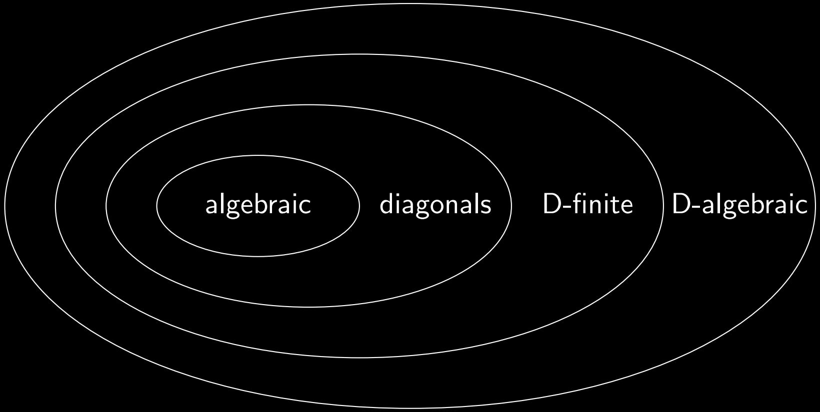
Papers related to D-finiteness at this year's ISSAC:

- **Louis Gaillard:** A unified approach for degree bound estimates of linear differential operators
- **Shaoshi Chen, Manuel Kauers, Christoph Koutschan, Xiuyun Li, Rong-Hua Wang and Yisen Wang:** Non-minimality of minimal telescopers explained by residues
- **Manuel Kauers and Raphael Pages:** Bounds for D-Algebraic Closure Properties
- **Alaa Ibrahim:** Positivity Proofs for Linear Recurrences with Several Dominant Eigenvalues
- **Jérémy Berthomieu, Romain Lebreton and Kevin Tran:** Quasi-Linear Guessing of Minimal Lexicographic Gröbner Bases of Ideals of C-Relations of Random Bi-Indexed Sequences


D-finite


algebraic
D-finite


algebraic
diagonals
D-finite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